

# Gap-Free Rigidity, Robust Certificates, and Chamberwise Global Cover/Atlas Universality for Extremal Growth in Finite Positive Templates

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## Abstract

We present a gap-free rigidity mechanism for extremal (minimal) growth in finite families of positive template operators. The core outcome is a finite certificate dichotomy: for every reduced descriptor instance, either one produces a concrete obstruction witness (OG1–OG4), or one produces an explicit Doeblin word and a recurrence modulus (RC1) that together yield uniform projective contraction along all tight trajectories. We then upgrade the pipeline to a robust form: each certificate is accompanied by explicit perturbation margins and a computed stability radius  $\varepsilon^*$  so the decision is stable under small changes of template entries. The resulting package yields a deterministic, checkable route from reduction to quantitative Hilbert-metric contraction, with constants that can be read off from the certificate itself. Finally, on compact normalized parameter families (within the certified class) we obtain a witness-producing global cover/atlas decider: either a finite set of robust certificates covers the region with a single uniform robustness radius and uniform rate pack (good-global contraction universality), or the procedure returns explicit obstruction/degeneracy data locating where and why uniform contraction cannot hold, packaged as a boundary-atlas report. All results are for finite positive templates in the certified regime and provide a mathematically controlled setting for effective long-run description via finite artifacts and robustness margins.

## 1 Overview

This paper is built for two complementary goals.

### Scope and non-claims

This work studies products of finitely many positive templates (finite-dimensional, order-preserving operators) under an explicit reduction/normalization contract. The results provide effective symbolic descriptors and witness-producing stability certificates for extremal growth dynamics on this certified class, together with perturbation margins and regionwise cover/atlas reports. The purpose is to give a verifiable, transportable mechanism for "effective description" in a mathematically controlled regime, rather than a general theory of turbulence for infinite-dimensional partial differential equations.

- The conclusions are stated for finite positive-template families and their induced tight-core boundary systems; no claim is made for unrestricted chaotic flows.
- The observables controlled are the extremal-growth projective dynamics (Hilbert-metric geometry on the active face) and the finite symbolic descriptors extracted from the tight core.

- The output is always finite and checkable: either quantitative contraction data with explicit constants and robustness radius, or a finite obstruction witness localizing the failure mode.

## Chaos-theory vocabulary bridge

Readers coming from chaos theory and dynamical systems often use broad terms (“effective description,” “predictability,” “turbulence,” “non-chaotic”), while this paper works with a certified finite-template class and produces explicit finite artifacts. The following dictionary fixes a common language so the scope and the deliverables are unambiguous.

| Chaos/dynamics phrase           | Meaning on the certified class in this paper                                                                                                                                                                                                                     |
|---------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Effective description           | A finite descriptor (tight-core boundary system + face data) together with a finite witness dichotomy: either an explicit obstruction witness (OG1–OG4), or an explicit Doeblin word plus recurrence data yielding a quantitative contraction rate.              |
| Predictability / loss of memory | Quantitative projective forgetting on the active face: a computable Hilbert-metric contraction rate in the witnessed “good” regime, and a localization of failure to explicit obstruction or margin-collapse loci otherwise.                                     |
| Statistical stability           | Robustness of both descriptors and rates under perturbations of template entries, packaged as margins and a computed stability radius $\varepsilon^*$ in the robust certificate objects (certificate contract through recurrence stability).                     |
| Non-chaotic behavior            | Uniform projective contraction on the minimizing set (a unique attracting projective section for extremal dynamics), with explicit decay rate and constants contained in the rate pack.                                                                          |
| Boundary / frontier of regimes  | A finite cover-or-atlas outcome on compact normalized families: either a finite robust cover with a uniform radius and rate pack, or a boundary-atlas report that returns explicit obstruction/degeneracy data indicating where uniform contraction cannot hold. |

**The finite artifacts produced by the pipeline.** Throughout, the point is that every conclusion is backed by a finite object that can be checked directly.

- **Descriptor and tight core:** a reduced tight subgraph  $G^{(0)}$  with SCC/periodicity labels and active-face data (§3, §6).
- **Obstruction witnesses:** explicit finite certificates OG1–OG4 locating the precise failure mode (§18, §36).
- **Doeblin word:** a concrete word whose product has full support on the active face, extracted by a finite search (§14).
- **Recurrence modulus:** a checkable recurrence predicate (RC1) that, together with Doeblin, yields witnessed contraction on the minimizing set (§15, §19).
- **Rate pack:** explicit constants and a computable exponential decay rate for projective contraction, derived from the witnessed Doeblin and recurrence data (§20).
- **Robust certificate:** margins and a computed stability radius  $\varepsilon^*$  ensuring the descriptor classification and the rate pack remain valid under small perturbations (certificate contract through recurrence stability).

- **Cover-or-atlas report:** on a compact normalized parameter family, a finite set of robust certificates giving a uniform “good-global” regime, or a boundary-atlas report returning explicit obstruction/degeneracy loci (§35 and §35.4).

**Minimal mental model.** A finite family of positive templates induces a reduced tight core capturing all extremal (minimal-growth) behavior. Either there is a finite obstruction witness (OG1–OG4), or an explicit Doeblin word on an aperiodic tight component together with a recurrence predicate (RC1) yields uniform projective contraction along all tight trajectories. Robust certificates attach explicit margins and a computed stability radius  $\epsilon^*$ , so the conclusion is stable under small perturbations. On any compact normalized parameter family  $\Theta$  within the certified class, this openness becomes a finite cover: the outcome is always *cover-or-atlas*—either a single uniform good-global contraction certificate, or a boundary-atlas report localizing the degeneracy where uniform contraction cannot hold.

- **Structural goal:** explain why “optimality forces regularity” in finite positive-template growth problems.
- **Operational goal:** give a finite certificate contract that either returns a specific obstruction (OG1–OG4) or returns quantitative contraction data that can be verified directly.

Technically, the contraction step is a quantitative use of Hilbert’s projective metric and Birkhoff’s contraction coefficient [11, 12, 13]. Conceptually, extremal itineraries are modeled by a finite-type shift and its sofic refinements [18], and the resulting rate packs interface naturally with the language of linear cocycles, random matrix products, and Lyapunov exponents [15, 16, 17]. The motivating growth viewpoint aligns with transfer-matrix product growth systems in statistical mechanics and related positive-operator settings [19].

## Related work: effective description in chaos and dynamics

A central ambition in chaos theory is to replace informal claims of “complexity” with effective objects that predict long-run behavior. For smooth hyperbolic systems this role is often played by *physical* (SRB) measures and their statistical properties, which provide a rigorous meaning of observable asymptotics for typical initial data [22]. In parallel, *symbolic dynamics* provides finite-state models (shifts of finite type and sofic shifts) that encode orbit structure and allow computable invariants [18].

The setting of this paper is different in two ways. First, the objects are finite-dimensional positive templates, so the long-run geometry of projective directions can be controlled quantitatively via Hilbert metric methods and Birkhoff-type contraction mechanisms when a Doeblin minorization is witnessed [11, 12, 13]. This viewpoint sits in the classical Perron–Frobenius tradition for primitive nonnegative matrices and Markov chains, where effective growth and mixing are controlled by positivity and minorization structure [14]. Second, the focus is extremal (minimal) growth, where the dynamics is naturally expressed through a reduced tight-core boundary system whose admissible itineraries form a finite-type language. This places the problem close to the interface of symbolic dynamics, linear cocycles, and products of positive operators (including transfer-matrix viewpoints), where Lyapunov exponents and growth rates are the principal observables [15, 16, 17, 19]. In statistical-mechanics language, transfer-matrix growth can also be organized through thermodynamic-formalism and transfer-operator perspectives [20], and cocycle viewpoints are naturally expressed in the framework of random dynamical systems [21].

From an “effective description” viewpoint, ergodic optimization and related extremal-selection principles ask for concrete structure of measures or itineraries that optimize a given functional. In broad smooth settings these problems can be subtle and are typically not decidable in any practical sense [23]. The contribution here is to identify a certified class where the extremal-growth boundary dynamics admits a finite artifact calculus: one either produces explicit obstruction witnesses (OG1–OG4) or produces quantitative contraction data with explicit constants and robustness margins. This places the work as a “decidable island” contribution in the chaos-and-dynamics landscape: a mathematically controlled regime where the effective objects (descriptors, witnesses, rate packs, and cover/atlas reports) can be computed and verified end-to-end.

The technical spine proceeds in three layers.

- **Reduction and tight core:** the descriptor reduction produces a tight subgraph  $G^{(0)}$  and local face data.
- **Gap-free forcing and contraction:** calibrated minimizers concentrate on the tight core; slack and return-cost arguments yield positive-density forcing; Doeblin and recurrence then give projective contraction on minimizers.
- **Certificates and robustness:** each step is packaged as a finite certificate with explicit robustness margins.

## 1.1 Contributions (artifact-keyed)

The contribution is not a single lemma but a package whose components are designed to be verifiable and reusable. Each item below is tied to a concrete artifact produced by the pipeline.

- **A finite boundary system for extremal growth.** Reduction and canonical normalization produce a tight-core boundary system (SFT/sofic refinements) that represents the extremal itineraries by finite state data. *Artifact:* tight-core graph  $G^{(0)}$  with SCC/periodicity labels and face certificates.
- **A witness-producing obstruction taxonomy.** When uniform contraction cannot be certified, the pipeline returns explicit finite witnesses (OG1–OG4) that localize the failure mechanism. The witnesses distinguish tie/degeneracy, absence of a Doeblin word, and recurrence avoidance. *Artifact:* `obstruction_witness.json` payloads, including explicit avoidance cycles in OG4.
- **A quantitative contraction mechanism with explicit constants.** On the aperiodic core, Doeblin minorization together with a bounded-gap recurrence predicate yields explicit Hilbert-metric contraction with a computable rate pack. *Artifact:* `rate_pack` object  $(\tau, C, \kappa)$ .
- **A robust certificate contract.** Every predicate used by the pipeline is packaged with explicit margins and a computable robustness radius  $\varepsilon^*$ , so conclusions are stable under perturbations with degraded constants given by closed formulas. *Artifact:* `robust_certificate.json` with margins  $(\gamma, \delta, \varepsilon^*)$ .
- **A cover-or-atlas decider on compact normalized families.** On any compact normalized parameter family  $\Theta$  within the certified class, region mode terminates with either a good-global contraction certificate (uniform margins and uniform rate pack) or a boundary-global atlas that localizes degeneracy neighborhoods. *Artifact:* `frontier_atlas_report.json` (standardized chamberwise frontier atlas summary), optionally together with the detailed audit objects `GlobalUniversalityReport` / `GlobalUniversalityCertificate`.

| Contribution             | Primary section(s) | Artifact handle(s)                                            |
|--------------------------|--------------------|---------------------------------------------------------------|
| Finite boundary system   | §3, §6             | tight core / face certificates                                |
| Obstruction taxonomy     | §18, §36           | OG1–OG4 witnesses                                             |
| Quantitative contraction | §19, §20           | rate pack $(\tau, C, \kappa)$                                 |
| Robustness contract      | §26–§29            | robust certificate + $\varepsilon^*$                          |
| Cover/atlas on $\Theta$  | §33, §35, §34.5    | <b>frontier_atlas_report.json</b><br>(+ optional report/cert) |

## Where to find what

| If you are looking for                                                                         | See                                                                                                                                                     |
|------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| Contribution map (artifact-keyed)                                                              | <a href="#">§1.1</a>                                                                                                                                    |
| Chaos-theory vocabulary bridge                                                                 | <a href="#">§1</a>                                                                                                                                      |
| Chaos-theory related work anchoring                                                            | <a href="#">§1</a>                                                                                                                                      |
| Decidable-island framing and scope limits                                                      | <a href="#">§1</a> , <a href="#">§2.5</a> , and <a href="#">§2.6</a>                                                                                    |
| Reduction, normalization, and the induced tight-core boundary system                           | <a href="#">§3</a> and <a href="#">§6</a> (with computation in <a href="#">§7</a> )                                                                     |
| Imprimitive tight cores (periodic trapping descriptor and cyclic classes)                      | <a href="#">§4</a>                                                                                                                                      |
| Obstruction taxonomy (OG1–OG4) and witness objects                                             | <a href="#">§18</a> and <a href="#">§36</a>                                                                                                             |
| Why Doeblin existence alone does not force contraction on minimizers (OG4 sharpness)           | <a href="#">§37</a> , <a href="#">Prop. 37.2</a>                                                                                                        |
| Doeblin-word decision and witnessed word extraction                                            | <a href="#">§14</a>                                                                                                                                     |
| Recurrence dichotomy, forcing, and syndetic recurrence tools                                   | <a href="#">§12</a> and <a href="#">§15</a>                                                                                                             |
| Contraction from Doeblin plus recurrence (including disjoint-occurrence counting in RH2)       | <a href="#">§19</a>                                                                                                                                     |
| Rate-pack extraction (constants, decay rates, and bounds)                                      | <a href="#">§20</a>                                                                                                                                     |
| Robustness radii and the stability layers (certificate contract through recurrence stability)  | <a href="#">§26</a> , <a href="#">§27</a> , <a href="#">§28</a> , <a href="#">§29</a>                                                                   |
| Statistical-stability analogue (descriptor stability + rate-pack continuity)                   | <a href="#">§30</a>                                                                                                                                     |
| Sensitive vs rough dependence analogue (predictability modulus + margin-collapse localization) | <a href="#">§31</a>                                                                                                                                     |
| Switching-control wedge (Doeblin success vs avoidance-cycle backfire)                          | <a href="#">§32</a>                                                                                                                                     |
| Global-on- $\Theta$ cover/atlas guarantee and output objects                                   | <a href="#">§33</a> , <a href="#">§34</a> , <a href="#">§35</a> , and <a href="#">§35.4</a> (with the $\Theta$ -family contract in <a href="#">§5</a> ) |
| Frontier atlas (chamber inequalities + boundary loci)                                          | <a href="#">§34.5</a>                                                                                                                                   |
| Algorithmic decidability and complexity notes                                                  | <a href="#">§2</a>                                                                                                                                      |
| Pipeline termination and soundness (witness dichotomy correctness)                             | <a href="#">§23</a>                                                                                                                                     |
| Conditional completeness (when uniform contraction forces a Doeblin block)                     | <a href="#">§24</a>                                                                                                                                     |
| Minimizer-level converse (contraction forces a finite recurrence modulus)                      | <a href="#">§25</a>                                                                                                                                     |

## 1.2 Main results at a glance

The paper contains several layers of results, but the core theorem chain is short. Each link below is stated in a witness-producing, checkable form, and each link has an explicit downstream artifact.

| Result                                    | What it guarantees (within the certified class)                                                                                                                                                          | Artifact output                                                                                                       | Where                             |
|-------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|-----------------------------------|
| Decidable island (RU-Stable)              | Either a robust contraction certificate (Doebelin + RC1 $\Rightarrow$ explicit projective contraction) or an explicit obstruction witness (OG1–OG4)                                                      | <code>robust_certificate.js</code> or<br><code>obstruction_witness.json</code> +<br><code>frontier_report.json</code> | Theorem 2.8                       |
| Gap-free contraction on tight dynamics    | Given an aperiodic tight SCC with a Doebelin witness, either OG4 holds (avoidance cycle) or else $w$ is unavoidable and yields uniform exponential Hilbert contraction on every tight path in the SCC    | RC1 modulus $L_{\text{syn}}$ , rate pack $(\tau, C, \kappa)$                                                          | Theorem 19.6                      |
| Contraction along minimizing trajectories | If a minimizer spends positive lower density of tight time in the aperiodic SCC, unavoidable recurrence forces positive lower density of Doebelin blocks, hence an explicit exponential contraction rate | Rate pack with density-forced $\kappa$                                                                                | Theorem 19.7                      |
| Robustness of the decision                | Every certificate field is equipped with explicit margins and a computed stability radius $\varepsilon^*$ , so the same decision (and degraded constants) holds under perturbations                      | Explicit $\varepsilon^*$ and degradation formulas                                                                     | Sections 26–29                    |
| Cover-or-atlas universality on $\Theta$   | On compact normalized $\Theta$ , either a finite set of robust neighborhoods covers $\Theta$ with a uniform rate pack, or else a boundary atlas localizes degeneracy/obstruction loci                    | <code>frontier_atlas_report.json</code> (+ optional re-port/cert)                                                     | Theorem 34.2 and Proposition 34.6 |

**Interpretation.** The first line is the central “decidable island” statement: rather than attempting to decide stability or chaos in full generality, we decide the presence of a robust, quantitative contraction mechanism by finite witnesses. The remaining lines record how the same finite artifacts propagate to explicit contraction rates and then to regionwise cover/atlas outputs.

## 1.3 A one-shot flagship theorem statement and proof map

The main results can be read as a single certified “effective description” theorem: on the certified class, the long-run extremal projective dynamics are either proven uniformly regular by a finite robust mechanism, or else a finite obstruction witness is returned that localizes the failure mode. On compact normalized parameter families, this dichotomy upgrades to a cover-or-atlas output.

**Theorem 1.1** (Certified effective description: witness dichotomy, explicit rates, and cover-or-atlas on compact families). *Consider an instance in the certified finite positive-template class of this paper (with the reduction/normalization contract in force). Then the following statements hold.*

- **Decidable island (instance level).** The pipeline terminates and returns exactly one finite artifact: either a robust contraction certificate establishing RU-Stable (Definition 2.7), or an explicit obstruction witness (OG1–OG4) (Theorem 2.8).
- **Gap-free contraction on the tight dynamics (rate pack).** If a robust contraction certificate is returned, it includes a tight aperiodic SCC  $H$ , a certified Doeblin word  $w$  on its face support, and the recurrence modulus (RC1). Consequently, the projective dynamics contract exponentially (Hilbert metric) along all tight trajectories in  $H$ , with an explicit rate pack  $(\tau, C, \kappa)$  computed from the certificate data (Theorem 19.6).
- **Contraction along minimizing trajectories.** On any calibrated minimizing trajectory whose tight time spends positive lower density in the aperiodic component, the RC1 forcing mechanism yields positive lower density of Doeblin blocks and therefore an explicit exponential contraction rate along the minimizer, with constants given by the same rate-pack outputs (Theorem 19.7).
- **Minimizer-level frontier sharpening.** If one posits a strong shift-uniform projective contraction property along the minimizing dynamics on an aperiodic tight SCC  $H$ , then a finite recurrence modulus for contractive blocks is forced: there exists  $L$  such that every length- $L$  block realized by minimizers is Doeblin ( $RC_{\min}^*$ ) (Proposition 25.5). Conversely, whenever  $RC_{\min}^*$  holds for some  $L$ , the minimizing dynamics on  $H$  are shift-uniformly contracting with explicit constants extracted from the finite minimizing palette (Proposition 25.7).
- **Robustness.** All predicates used above are equipped with explicit margins and a computed robustness radius  $\varepsilon^*$ . For perturbations within this radius (in the perturbation metric recorded by the robust certificate contract), the same decision outcome holds, and the contraction constants degrade by explicit formulas (Sections 26–29).
- **Cover-or-atlas on compact normalized families.** Let  $\Theta$  be a compact normalized parameter family within the certified class. Then the global procedure terminates with a finite output: either a good-global contraction certificate on  $\Theta$  with uniform margins and a uniform rate pack (Theorem 34.2 and Proposition 34.6), or a boundary-atlas report localizing near-degeneracy/obstruction loci.

*Proof map.* The instance-level dichotomy is Theorem 2.8. Given a Doeblin witness on an aperiodic tight SCC, the finite recurrence dichotomy and Hilbert-metric contraction with explicit constants are proved in Theorem 19.6, and the density-forcing upgrade along calibrated minimizers is Theorem 19.7. The minimizer-level recurrence converse and its forward closure are Propositions 25.5 and 25.7. Robustness of the certificate fields and explicit stability radii are the content of the RU-layer results (Sections 26–29). On a compact normalized family  $\Theta$ , uniform-margin global contraction is Theorem 34.2; region-mode coverage yields a good-global contraction certificate by Proposition 34.6; and the hybrid cover/atlas procedure is Algorithm 9.  $\square$

## 1.4 End-to-end certificate pipeline and artifact bundle

The results are designed to be consumed as a finite-output pipeline. At the level of a single instance, the output is either an obstruction witness (OG1–OG4) or a robust contraction certificate with explicit rate data. On a compact normalized family  $\Theta$ , the output upgrades to a cover-or-atlas report.

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**Algorithm 1** End-to-end certificate pipeline (instance mode and optional  $\Theta$ -mode)

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**Require:** Positive-template family (and costs, if present) satisfying the certified-class contract

**Require:** Perturbation metric and norms (robust certificate contract)

**Ensure:** A primary witness/certificate artifact (`obstruction_witness.json` or `robust_certificate.json`), and a compact `frontier_report.json` summary; optionally a `frontier_atlas_report.json` (and, when fully certified, a compressed region certificate) on compact normalized  $\Theta$

- 1: Canonical normalization and potential reduction (Section 3)
  - 2: Compute reduced costs, extract tight edges, build tight-core graph  $G^{(0)}$  and SCC data (Sections 6 and 7)
  - 3: Identify candidate aperiodic tight SCC(s)  $H$  and their face supports  $S(H)$
  - 4: Search for a Doeblin word  $w$  on  $S(H)$  (Section 14)
  - 5: **if** no Doeblin word exists on all aperiodic tight SCCs **then**
  - 6:     Return obstruction witness OG2/OG3 (explicit failure data), optionally serialized as `og2_inconsistency_witness.json` or `og3_cut_witness.json/og3_nonreachability_certificate.json`
  - 7: **else**
  - 8:     Build avoidance automaton for  $(H, w)$  and search for an avoidance cycle (Section 18)
  - 9:     **if** avoidance cycle found **then**
  - 10:         Return obstruction witness OG4 (explicit directed cycle)
  - 11:     **else**
  - 12:         Compute recurrence modulus (RC1) and forcing metadata (Sections 12 and 15)
  - 13:         Compute explicit rate pack  $(\tau, C, \kappa)$  from Doeblin+recurrence (Sections 19 and 20)
  - 14:         Compute robustness margins  $(\gamma, \delta)$  and stability radius  $\varepsilon^*$  with degradation formulas (Sections 26–29)
  - 15:         Return `robust_certificate.json` with witness data, `rate_pack`, and  $\varepsilon^*$ , together with `frontier_report.json`
  - 16:     **end if**
  - 17: **end if**
  - 18: **if** a compact normalized parameter family  $\Theta$  is provided **then**
  - 19:     Run the hybrid cover/atlas check (Algorithm 9)
  - 20:     Return `frontier_atlas_report.json` encoding the chamberwise labeling and declared frontier loci;
  - 21:     Optionally also return `GlobalUniversalityCertificate/GlobalUniversalityReport` as a more detailed audit object.
  - 22: **end if**
- 

The algorithm above is a reading guide rather than an additional hypothesis: each step corresponds to a stated computation or to a witnessed decision in the body of the paper.

## 2 Algorithmic decidability and complexity notes

The core deliverable of this paper is a finite, witness-producing decision procedure on the certified class. This section records conservative complexity bounds for the main subroutines, phrased in the natural graph parameters of the reduced descriptor. The goal is to make the “decidable island” claim operational: the pipeline is not merely existential, but mechanically checkable with transparent computational cost.

## 2.1 Input size parameters

We use the following standard size parameters.

- $n := |V|$  is the number of vertices in the reduced descriptor graph (or the relevant tight-core subgraph).
- $m := |E|$  is the number of directed edges in that graph.
- $d$  is the ambient dimension of the positive templates.
- $k$  is the number of states in the active tight component on which the Doeblin search is performed.
- $\ell$  denotes a word length bound used in bounded searches.

All costs below are worst-case upper bounds and can be improved in concrete implementations.

## 2.2 Main subroutines

**Proposition 2.1** (Finite termination of the certificate pipeline). *Fix a reduced descriptor instance satisfying the certificate pipeline contract. The pipeline terminates after finitely many elementary graph computations and returns either an obstruction witness (OG1–OG4) or a quantitative rate pack with a witnessed Doeblin word and a verified recurrence predicate.*

*Proof.* Each step is a finite procedure on finite combinatorial objects. Tight-core extraction, SCC decomposition, and periodicity labeling are finite-time graph operations. The Doeblin-word decision is posed as reachability in a finite pattern graph (constructed from the induced face dynamics), and the recurrence predicate is a bounded-gap verification on the same finite state space. Thus the pipeline halts with either a witness of failure or a witness of success.  $\square$

**Proposition 2.2** (Conservative complexity bounds for the core graph layer). *The following costs hold for a descriptor graph with  $n$  vertices and  $m$  edges.*

- *Minimum cycle-mean and canonical potentials (Karp + Bellman–Ford):  $O(nm)$ .*
- *Tight-edge extraction and SCC decomposition:  $O(n + m)$ .*
- *Aperiodicity labeling via gcd-of-cycle-lengths on each SCC:  $O(n + m)$ .*

*Proof.* The bounds are standard: Karp’s minimum mean cycle algorithm runs in  $O(nm)$ , and Bellman–Ford runs in  $O(nm)$ . SCC decomposition (Tarjan/Kosaraju) is linear time. The gcd-of-cycle-lengths computation can be implemented with one traversal per SCC using level differences and gcd updates, hence linear in the SCC size.  $\square$

**Proposition 2.3** (Conservative complexity bounds for Doeblin and recurrence verification). *Let  $k$  be the size of the aperiodic tight component used for Doeblin search.*

- *Doeblin-word search as reachability in the Boolean-pattern semigroup has worst-case state space size at most  $2^k$  (subset patterns) or  $2^{k^2}$  (Boolean matrices), depending on the chosen representation. A BFS reachability search therefore has worst-case cost  $O(|\mathcal{S}| \cdot s)$  where  $|\mathcal{S}|$  is the explored pattern set and  $s$  is the alphabet size.*

- Given a witnessed Doeblin word of length  $\ell$ , verifying the bounded-gap recurrence predicate on the same finite pattern graph has cost  $O((n + m)\ell)$  in the natural automaton implementation.

**Remark 2.4** (Why these bounds are acceptable in the certified regime). *The certified class is designed so that the active tight component and face dynamics remain small and structurally stable once margins are positive. The pipeline is intended to be used in regimes where  $k$  is moderate (and often far smaller than the ambient parameter discretizations), so the exponential worst-case bounds in  $k$  are not the practical bottleneck. The role of the present propositions is to make explicit where combinatorial blow-up can occur and how it is controlled by the reduction and certification constraints.*

### 2.3 Global-on- $\Theta$ complexity

In region mode, the global cover/atlas procedure evaluates the instance-mode pipeline on a finite  $\varepsilon$ -net of the compact normalized parameter family  $\Theta$ .

**Proposition 2.5** (Global cover/atlas work bound). *Let  $N_{\text{net}}(\Theta, \varepsilon)$  be the size of an  $\varepsilon$ -net used for the cover/atlas verification step. Then the global procedure terminates after at most  $N_{\text{net}}(\Theta, \varepsilon)$  invocations of the instance-mode pipeline plus the overhead needed to merge certificates into a cover or to assemble the boundary atlas.*

**Remark 2.6.** *The explicit net-size bounds in §35 therefore translate directly into conservative bounds on the total verification workload for the global universality decider.*

### 2.4 Referee map and verification checklist

This subsection is a navigation aid for referees and readers who want to audit the logical spine and the “finite artifact” outputs without reading every narrative paragraph. It does not introduce new claims; it points to where each claim is proved and to which finite outputs realize it.

**Proof spine (what implies what).** The flagship statement is Theorem 1.1. Its six bullets are proved by the theorems and algorithms listed in Table 1.

**Fast audit checklist (artifact-first).** A reader can sanity-check the entire program by following the artifact contracts rather than the narrative. The checklist below is phrased so each item can be verified by reading a single definition/lemma plus its associated algorithm.

- **Reduction contract.** Confirm the normalization/reduction assumptions and the tightness potential construction (Sections 3–7).
- **Tight core and periodicity.** Verify SCC decomposition and aperiodicity labels are computed from the tight graph and are stable under a strict tightness gap (Section 27).
- **Doeblin witness.** Verify the product-automaton search produces a concrete word  $w$  and a face support certificate, or else returns an explicit failure witness (Section 14 and Section 28).
- **OG2/OG3 witness format.** Verify that face-map failure and Doeblin failure can be serialized as a finite inconsistency-cycle witness and either a cut witness or a product-automaton nonreachability certificate (Lemma 13.9, Lemma 13.10, Corollary 13.6).

| Claim in Theorem 1.1                      | Load-bearing<br>ment(s)                          | state- | Primary file(s)                                                                                                                     | Artifact output(s)                                                                           |
|-------------------------------------------|--------------------------------------------------|--------|-------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------|
| Decidable island (instance level)         | Theorem 2.8                                      |        | jsr_decidability_frontier                                                                                                           | robust_certificate.json<br>or<br>robust_certificate.json<br>+<br>frontier_report.json        |
| Gap-free contraction on tight dynamics    | Theorem 19.6 and rate formulas in §20            |        | contraction_from_doeblin_rate_pack.tex                                                                                              | rate_pack_fields                                                                             |
| Contraction along minimizing trajectories | Theorem 19.7 and the forcing chain               |        | forcing_positive_density_rate_pack.tex                                                                                              | rate_pack_reused                                                                             |
| Minimizer-level frontier sharpening       | Propositions 25.5 and 25.7                       |        | contraction_from_doeblin_along_minimizers<br>minimizer_level_recurrence_converse_palette.tex                                        | minimizers<br>recurrence_converse_palette_constants<br>( $L, \varepsilon_{\min}, U_{\max}$ ) |
| Robustness radius and degraded constants  | Sections 26–29                                   |        | robust_certificate_scheme_star, de-<br>tight_core_stability.tex, degraded<br>doebelin_stability.tex,<br>recurrence_robustness.tex   | robust_certificate_scheme_star, de-<br>tight_core_stability.tex, degraded<br>( $\tau', C'$ ) |
| Cover-or-atlas on compact families        | Theorem 34.2, Proposi-<br>tion 34.6, Algorithm 9 |        | global_universality_theorem_atlas_report.json<br>region_net_certification.tex, optionally<br>global_universality_compact_minimizers | cover_or_atlas_report.json<br>(an)tex, optionally<br>GlobalUniversalityCertificate           |

Table 1: Referee map: where each flagship claim is proved and which finite output object realizes it.

- **Pipeline correctness.** Verify the termination and soundness theorem that ties each branch to its witness lemma and the contraction branch to the Doeblin + recurrence contraction theorem (Section 23).
- **Conditional completeness on tight dynamics.** Verify that uniform projective contraction on the induced tight subshift forces the existence of a Doeblin block and excludes avoidance cycles, yielding an equivalence between “uniform contraction” and “Doeblin + non-avoidance” on a tight SCC (Section 24).
- **Minimizer-level converse and closure.** Verify that a strong, shift-uniform form of trajectory-level contraction forces a finite recurrence modulus for contractive blocks along minimizers ( $\text{RC}_{\min}^*$ ), and that this finite palette condition yields shift-uniform contraction with explicit constants extracted from the palette (Section 25, Propositions 25.5 and 25.7).
- **Recurrence predicate.** Verify how (RC1) is checked or refuted via the avoidance graph, and how this converts to a bounded-gap return modulus (Section 15 and Section 29).
- **Rate pack.** Verify the explicit contraction constant chain from Doeblin minorization to Hilbert-metric contraction, and the closed-form rate-pack extraction (Sections 19 and 20).
- **Robustness.** Verify that each predicate used in the pipeline is recorded with explicit margins and that the stability radius  $\varepsilon^*$  is computed from those margins (Section 26).
- **Global-on- $\Theta$ .** Verify the net/region procedure and the cover-or-atlas termination statement, including the explicit work bound via the net size (Sections 34–35).

- **Necessity of the obstruction taxonomy.** Confirm that each obstruction class OG1–OG4 is required by a concrete realizable instance that passes all earlier gates (Section 36, Proposition 36.1).
- **Sharpness of the recurrence layer.** Verify that Doeblin existence does not imply contraction on all minimizers without a recurrence/forcing hypothesis (Section 37, Proposition 37.2).
- **Decidability and work bounds.** Confirm the conservative complexity bounds for each subroutine and the global net-based work bound (Section 2).

**What “machine-checkable” means here.** The term is used in the limited sense that every decision branch is represented by a finite artifact and every quantitative claim in the certified regime reduces to checking finitely many inequalities whose inputs are recorded in that artifact. The included JSON schemas specify the required fields and their semantics for the witness and certificate objects (`frontier_report_schema.json`, `frontier_atlas_report_schema.json`, `obstruction_witness_schema.json`, `og2_inconsistency_witness_schema.json`, `og3_cut_witness_schema.json`, `og3_nonreachability_certificate_schema.json`, `rate_pack_schema.json`, `robust_certificate_schema.json`, `global_universality_report_schema.json`, and `global_universality_certificate_schema.json`).

**Compilation.** The LaTeX entry point is `rigidity_gapfree_global_universality.tex`. A standard build is `latexmk -pdf` on that file (or two runs of `pdflatex`).

**Schema validation.** Schema files (JSON Schema draft 2020-12) are included for the witness and certificate artifacts. Validation is optional and can be performed with any standards-compliant validator; for example, one may run `python -m jsonschema -i <file.json> <schema.json>` for the included `GlobalUniversalityReport`, `GlobalUniversalityCertificate`, `RobustCertificate`, `ObstructionWitness`, and `RatePack` objects.

## 2.5 Decidable island for robust switching stability

In the general theory of switching systems and joint spectral radius (JSR), many natural decision problems are not decidable in full generality. For instance, even boundedness questions for products of a fixed finite matrix set can be undecidable in broad rational classes [1]. Accordingly, progress on “where stability is decidable” typically takes the form of identifying structurally constrained subclasses in which stability reduces to finite witnesses.

The certificate program developed here provides such a subclass. Rather than attempting to decide *JSR* in complete generality, we decide a stronger, certificate-based property: whether the instance admits a *robust uniform contraction mechanism* that is both checkable and stable under perturbation.

**Definition 2.7** (Robust contraction certificate property). *Fix a perturbation model and norm as in Section 26. An instance satisfies RU-Stable if the pipeline returns a contraction certificate whose fields include:*

- a tight aperiodic SCC  $H$  on which the tight-core data are valid,
- a Doeblin word witness  $w$  with margin  $\delta > 0$  on the SCC face support,
- a recurrence modulus (RC1) for  $(H, w)$ , yielding bounded-gap recurrence of  $w$  along every tight minimizer in  $H$ ,
- an explicit projective contraction-rate pack  $(\tau, C, \kappa)$ ,

- an explicit robustness radius  $\varepsilon^* > 0$  such that all of the above remain valid (with degraded constants) for perturbations of size at most  $\varepsilon^*$ .

**Theorem 2.8** (Decidability frontier in the certified class). *For every instance in the certified finite positive-template class considered in this paper, the pipeline terminates and returns exactly one of the following finite, verifiable artifacts.*

- A finite obstruction witness (*OG1–OG4*), certifying that the data required by Definition 2.7 cannot be obtained from the instance by the certificate rules. In particular, *OG4* returns an explicit directed cycle in the avoidance automaton, producing a tight periodic trajectory that avoids the Doeblin word forever.
- A robust contraction certificate establishing RU-Stable for the instance, including an explicit robustness radius  $\varepsilon^*$  and explicit rate degradation formulas.

Consequently, membership in the property RU-Stable is decidable on this class.

**Remark 2.9** (Relation to JSR and switching stability). *Theorem 2.8 does not claim a decision procedure for JSR in unrestricted classes. Instead, it identifies a concrete decidable island: within the present positive-template setting, one can decide whether a robust, quantitative uniform contraction mechanism exists, and one can locate precisely where this mechanism fails via explicit witnesses. This is consistent with, and complementary to, general undecidability barriers [1].*

## 2.6 Resolution map for neighboring open-problem families

Several neighboring research programs study joint spectral radius, extremal products, ergodic optimization, and synchronization phenomena in broad generality. In full generality many decision questions are provably undecidable, and many structural questions are answered only at the level of existence or genericity. The present work contributes a complementary kind of result: a *certificate calculus* that turns a range of these themes into finite objects whose validity is decidable on the certified positive-template class.

The table below records, for each commonly cited open-problem family, the concrete statement resolved *within the certified class* and the pipeline artifact that realizes the statement.

**Interpretation.** The point is not to decide joint spectral radius in unrestricted families. Instead, the paper identifies a structurally defined subclass in which the relevant stability and extremal-structure conclusions reduce to finite, checkable certificates. The certificate margins then make these conclusions robust and enable region certification.

**Chaos-theory positioning: “decidable islands” as effective description.** Li’s list of major open problems in chaos theory emphasizes a gap between qualitative narratives (“chaos”/“turbulence”) and *effective structure* that can be computed, verified, and transported across parameters [5]. Within the certified positive-template class, the deliverable is precisely such effective structure: a finite symbolic descriptor (tight-core language plus finite refinements) together with a witness-producing dichotomy that either yields quantitative stability (explicit contraction data) or returns explicit finite obstructions localizing where stability cannot hold. The robustness margins  $(\gamma, \delta, \varepsilon^*)$  then turn this into a chamberwise and regionwise “frontier atlas” that separates stable regimes from boundary pathology by checkable inequalities.

| Open-problem family                                                          | Resolved statement in the certified class                                                                                                                                                                                                                                                                                                                           | Pipeline artifact                                                   |
|------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| Decidability of robust switching stability / JSR-type questions [1, 3]       | Membership in RU-Stable is decidable: the pipeline returns either a robust contraction certificate (with $\varepsilon^*$ and explicit rate degradation) or an explicit obstruction witness (OG1–OG4), including the avoidance-cycle witness for recurrence failure (OG4).                                                                                           | <code>robust_certificate</code> or <code>obstruction_witness</code> |
| Mather-set structure for matrix sequences (ergodic optimization for JSR) [2] | The extremal set is represented effectively by a finite symbolic system: a tight-core subshift (Definition 6.6) refined by the avoidance/recurrence automaton (Section 15), yielding an explicit sofic description of extremal itineraries in the certified regime.                                                                                                 | Tight-core graph $H$ + avoidance automaton                          |
| Generic finiteness behavior for JSR and chamber regularity [4]               | Away from certificate-failure surfaces (notably $\gamma \rightarrow 0$ and $\delta \rightarrow 0$ ) the tight core and obstruction regime are locally constant by the tight-core and Doeblin stability theorems, so extremal combinatorics and rates are stable on chambers. Within a chamber, periodicity and recurrence are decided by the obstruction dichotomy. | $\gamma, \delta, \varepsilon^*$ + chamber report                    |
| Devil’s-staircase boundary complexity vs. tame regions [4]                   | Complex behavior is confined to explicit boundary loci where the certificate margins collapse. The robustness layer identifies and localizes these loci, separating stable chambers from boundary pathology by explicit, checkable inequalities.                                                                                                                    | Exceptional-set report (region mode)                                |

Table 2: How the robust certificate program resolves several open-problem families within the certified positive-template class (part 1).

**Decidability frontier for robust switching stability.** Theorem 2.8 gives a dichotomy: either a robust certificate exists, producing explicit contraction rates and a robustness radius  $\varepsilon^*$  (Definition 26.6), or an obstruction witness exists (OG1–OG4), and the witness is finite and verifiable. This realizes a decidable island inside the broader JSR landscape where global decision problems are known to be hard and can be undecidable [1].

**Effective Mather-set and extremal-language description.** Within the certified class, the tight-core subshift (Definition 6.6) and the avoidance/recurrence automata (Section 15) produce an explicit sofic model for extremal itineraries. This is the finite-state analogue of the qualitative Mather-set structure developed in the ergodic-optimization literature [2].

**Generic periodicity and chamber stability.** The chamber machinery (Section 38) states that, away from explicit tie and margin-collapse loci, the tight combinatorics and the obstruction regime are locally constant. In those chambers, the pipeline decides whether extremals are periodic (unique tight cycle) or genuinely aperiodic (tight aperiodic SCC with certified recurrence), and it produces the corresponding finite model.

**Devil’s-staircase boundaries as certificate-failure surfaces.** The region certification layer (Section 33) turns the informal slogan “complexity lives on the boundary” into a checkable statement: nontrivial parameter dependence can only occur where the robustness margins  $(\gamma, \delta, \varepsilon^*)$  collapse. This separates stable chambers from boundary pathology in a way that complements known devil’s-staircase phenomena in broader families [4].

**Synchronizing bounds for induced automata.** The Doeblin search automaton (OG3) and the avoidance/recurrence automaton (OG4) are built explicitly and come with finite state counts. In the acyclic-avoidance regime, the return modulus  $B$  is computed (Algorithm 8), which is already a quantitative recurrence statement. When the induced machines fall into tractable synchronizing subclasses (aperiodic, trivial group in the transition monoid, or related checkable hypotheses), known Černý-type bounds become available for these induced automata [6, 7].

**Quantitative bounds on Doeblin-word length.** When a Doeblin block exists, the pipeline produces an explicit positive word and margin  $\delta$  (OG3), and the robustness layer converts  $\delta$  into contraction data. In the certified setting, standard exponent bounds for primitive directed graphs can be used as plug-in upper bounds for the worst-case search radius on the core support [8].

**Constructive extremal geometry and extremal-norm analogues.** The calibrated section and rate pack (Section 20) provide a concrete geometric object that governs the extremal dynamics together with quantitative contraction and robustness radii. This aligns with the extremal-norm narrative around Barabanov norms, but in the certified class the object is produced and verified by finite witnesses [9].

**Bounded-language semigroup decidability template.** The certified language of admissible products is not arbitrary: it is the tight-core language refined by explicit avoidance constraints. This is precisely the kind of language restriction under which semigroup problems can become decidable in settings where the unrestricted problems are intractable [10].

**Frontier mapping and explicit witnesses.** Finally, the obstruction taxonomy (OG1–OG4) and the robustness margins provide an operational map of why certification fails, returning concrete finite witnesses rather than leaving failure as an opaque non-result. This is the main deliverable that makes the subclass practically usable: it either certifies stability with quantitative control or explains non-certifiability with an explicit obstruction.

**Toward cover/atlas universality on compact normalized families within the certified class.** Theorem 34.2 isolates the remaining step needed to upgrade from local robustness to a global statement on a compact region  $\Theta$ : prove that the named margin-collapse loci are absent on  $\Theta$  (equivalently: obtain uniform positive lower bounds for the certificate margins). This can be achieved either by structural hypotheses that enforce uniform gaps, or by region mode plus a finite covering argument that certifies  $\Theta$  and produces a global robustness radius. If  $\Theta$  intersects the named margin-collapse loci, the global procedure still terminates with a boundary-atlas report that localizes the degeneracy/obstruction region rather than asserting uniform contraction.

### 3 Canonical Normalization and the Tight Subgraph

This section records the structural normalization that underlies the rest of the paper. It converts the extremal-growth problem into the study of a canonically defined zero-set (the *tight* transitions). Subsequent rigidity and obstruction statements are properties of this tight geometry.

#### 3.1 Reduced costs

Let  $G = (V, E)$  be the reduced finite graph produced by the descriptor reduction, with edge costs  $c : E \rightarrow \mathbb{R}$ . Define the minimal cycle mean

$$\lambda_* := \min_{\gamma \text{ cycle}} \frac{1}{|\gamma|} \sum_{e \in \gamma} c(e).$$

**Theorem 3.1** (Canonical normalization by a potential). *There exists a function (potential)  $\phi : V \rightarrow \mathbb{R}$  such that the reduced costs*

$$c_\phi(e) := c(e) - \lambda_* + \phi(\text{tail}(e)) - \phi(\text{head}(e))$$

*satisfy  $c_\phi(e) \geq 0$  for all  $e \in E$ .*

*Proof.* Let  $\tilde{c}(e) := c(e) - \lambda_*$ . For any directed cycle  $\gamma$  of length  $k$ ,  $\sum_{e \in \gamma} \tilde{c}(e) = k(\bar{c}(\gamma) - \lambda_*) \geq 0$ , so  $(G, \tilde{c})$  has no negative cycles. Fix a root  $v_0$  in each reachable component and define  $\phi(v)$  to be the minimum  $\tilde{c}$ -cost of a directed path from  $v_0$  to  $v$ . Then for any edge  $e : u \rightarrow v$ , optimality of  $\phi(v)$  yields  $\phi(v) \leq \phi(u) + \tilde{c}(e)$ , i.e.  $\tilde{c}(e) + \phi(u) - \phi(v) \geq 0$ . This inequality is exactly  $c_\phi(e) \geq 0$ .  $\square$

#### 3.2 The tight subgraph

**Definition 3.2** (Tight edges and tight subgraph). *Fix a potential  $\phi$  as in Theorem 3.1. Define the tight edges*

$$E^{(0)} := \{e \in E : c_\phi(e) = 0\}, \quad G^{(0)} := (V^{(0)}, E^{(0)}),$$

*where  $V^{(0)}$  are vertices incident to  $E^{(0)}$ .*

#### 3.3 Period and aperiodicity of a tight SCC

**Definition 3.3** (Period of a strongly connected digraph). *Let  $H$  be a strongly connected directed graph. Its period is*

$$\text{per}(H) := \gcd\{|\gamma| : \gamma \text{ is a directed cycle in } H\}.$$

*We call  $H$  aperiodic if  $\text{per}(H) = 1$ , and imprimitive otherwise.*

#### 3.4 Minimizers are asymptotically tight

A (one-sided) path  $\pi = (e_0, e_1, \dots)$  has length- $n$  average cost

$$\text{Avg}_n(\pi) := \frac{1}{n} \sum_{t=0}^{n-1} c(e_t).$$

We call  $\pi$  *minimizing* if  $\liminf_{n \rightarrow \infty} \text{Avg}_n(\pi) = \lambda_*$ .

**Proposition 3.4** (Asymptotic tightness of minimizing paths). *Let  $\pi$  be a minimizing path and fix  $\phi$  as above. Then the set of times  $t$  with  $e_t \notin E^{(0)}$  has asymptotic density 0. Equivalently,  $E^{(0)}$  is visited with asymptotic density 1 along  $\pi$ .*

*Proof.* Because  $E$  is finite and  $c_\phi(e) \geq 0$ , define

$$\alpha := \min\{c_\phi(e) : e \in E \setminus E^{(0)}\},$$

with the convention  $\alpha = +\infty$  if  $E^{(0)} = E$ . When  $E^{(0)} \neq E$ , we have  $\alpha > 0$ . Let  $N_n$  be the number of indices  $t < n$  with  $e_t \notin E^{(0)}$ . Then  $\sum_{t=0}^{n-1} c_\phi(e_t) \geq \alpha N_n$ . On the other hand, telescoping the potential gives

$$\sum_{t=0}^{n-1} c_\phi(e_t) = \sum_{t=0}^{n-1} (c(e_t) - \lambda_*) + \phi(v_0) - \phi(v_n),$$

where  $v_n$  is the vertex reached after  $n$  steps. Since  $\phi$  is bounded on the finite reachable vertex set,  $|\phi(v_0) - \phi(v_n)| \leq C$ . Divide by  $n$  and take  $\liminf$ . Minimizing implies  $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} (c(e_t) - \lambda_*) = 0$ , hence  $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} c_\phi(e_t) = 0$ . Therefore  $\liminf_{n \rightarrow \infty} \alpha N_n/n = 0$ , which forces  $N_n/n \rightarrow 0$ .  $\square$

### 3.5 A tight-core dichotomy

For a subset  $A \subseteq \mathbb{N}$ , write its *lower asymptotic density* as

$$\underline{d}(A) := \liminf_{n \rightarrow \infty} \frac{|A \cap \{0, 1, \dots, n-1\}|}{n}.$$

**Proposition 3.5** (Tight-core dichotomy). *Let  $\pi$  be a minimizing path and let  $T := \{t \geq 0 : e_t \in E^{(0)}\}$  be its set of tight times. For each SCC  $H$  of the tight graph  $G^{(0)}$ , set*

$$T_H := \{t \in T : v_t \in H\},$$

*where  $v_t$  is the tail vertex of  $e_t$ . Then there exists at least one SCC  $H$  with  $\underline{d}(T_H) > 0$ . Moreover, exactly one of the following holds:*

(P) *every SCC  $H$  with  $\underline{d}(T_H) > 0$  is imprimitive, i.e.  $\text{per}(H) > 1$  (Definition 3.3);*

(A) *there exists an aperiodic SCC  $H$  of  $G^{(0)}$  with  $\underline{d}(T_H) > 0$ , i.e.  $\text{per}(H) = 1$ .*

*Proof.* Let  $T := \{t \geq 0 : e_t \in E^{(0)}\}$  be the set of *tight times* along  $\pi$ . By Proposition 3.4,  $T$  has asymptotic density 1, hence lower density 1. For each strongly connected component  $H$  of the tight graph  $G^{(0)}$ , set

$$T_H := \{t \in T : \text{the time-}t \text{ vertex } v_t \text{ lies in } H\}.$$

The sets  $(T_H)_H$  form a *finite* partition of  $T$  indexed by the SCCs of  $G^{(0)}$ . If every  $T_H$  had lower density 0, then their finite union  $T$  would also have lower density 0, contradicting  $\underline{d}(T) = 1$ . Therefore there exists at least one SCC  $H$  with  $\underline{d}(T_H) > 0$ .

If among the SCCs with  $\underline{d}(T_H) > 0$  there is an aperiodic one, we are in case (A). Otherwise every SCC visited with positive lower density along tight times is imprimitive, which is case (P).  $\square$

**Remark 3.6.** *The remainder of the paper analyzes the geometry of  $G^{(0)}$ . Rigidity (projective contraction) is obtained from recurrence and Doeblin structure inside an aperiodic tight SCC. Obstructions are finite failures of the corresponding tight-core regularity conditions.*

## 4 Imprimitive tight cores: cyclic class decomposition

In the imprimitive branch (OG1), the correct output is not a contraction claim but a *periodic trapping descriptor*. An imprimitive strongly connected component has an intrinsic period  $p > 1$  and a canonical partition into  $p$  cyclic classes. This provides an effective, finite description of the tight dynamics: tight trajectories must rotate through these classes.

**Definition 4.1** (Cyclic class decomposition of an SCC). *Let  $H$  be a strongly connected directed graph with period  $p := \text{per}(H)$  (Definition 3.3). Fix a root vertex  $r \in V(H)$ . For each vertex  $v \in V(H)$ , define*

$$\text{cls}(v) := (\ell(r \rightarrow v) \bmod p) \in \{0, 1, \dots, p-1\},$$

where  $\ell(r \rightarrow v)$  denotes the length of any directed path from  $r$  to  $v$ . For  $k \in \{0, 1, \dots, p-1\}$  define the cyclic class

$$C_k := \{v \in V(H) : \text{cls}(v) = k\}.$$

**Lemma 4.2** (Well-definedness and rotation property). *The class label  $\text{cls}(v)$  in Definition 4.1 is well-defined (independent of the chosen path  $r \rightarrow v$ ). Moreover, for every directed edge  $u \rightarrow v$  in  $H$  one has*

$$\text{cls}(v) \equiv \text{cls}(u) + 1 \pmod{p}.$$

Consequently, the sets  $C_0, \dots, C_{p-1}$  form a partition of  $V(H)$  and every directed edge of  $H$  goes from  $C_k$  to  $C_{k+1 \bmod p}$ .

*Proof.* Let  $\gamma_1$  and  $\gamma_2$  be two directed paths from  $r$  to  $v$  with lengths  $\ell_1$  and  $\ell_2$ . Because  $H$  is strongly connected, there exists a directed path  $\eta$  from  $v$  back to  $r$  with length  $m$ . Then  $\gamma_1\eta$  and  $\gamma_2\eta$  are directed cycles in  $H$  with lengths  $\ell_1 + m$  and  $\ell_2 + m$ . By definition of  $p = \text{per}(H)$  as the gcd of cycle lengths, both  $\ell_1 + m$  and  $\ell_2 + m$  are divisible by  $p$ . Subtracting shows  $\ell_1 - \ell_2$  is divisible by  $p$ , hence  $\ell_1 \equiv \ell_2 \pmod{p}$ . This proves well-definedness.

For an edge  $u \rightarrow v$ , choose a path  $r \rightarrow u$  of length  $\ell$ . Appending the edge gives a path  $r \rightarrow v$  of length  $\ell + 1$ , so  $\text{cls}(v) \equiv \text{cls}(u) + 1 \pmod{p}$ . The partition and rotation statement follow immediately.  $\square$

**Remark 4.3** (Finite witness content in the OG1 branch). *Given a tight SCC  $H$ , the pipeline already computes  $p = \text{per}(H)$ . Lemma 4.2 shows that one may additionally output the cyclic partition  $\{C_k\}_{k=0}^{p-1}$  as a finite descriptor. In particular, any tight path supported in  $H$  must visit the classes in cyclic order; this is a concrete, checkable form of “periodic trapping”.*

## 5 A concrete compact normalized target family $\Theta_{\text{can}}$ (ready-to-run)

The global machinery of Sections 34–35 applies to any compact normalized parameter family. To remove ambiguity about what “choose  $\Theta$ ” means in practice, we record a canonical, concrete choice that is broad enough to cover most finite-template models while remaining fully compatible with the robust-certificate contract.

### 5.1 Raw parameters and normalization

Fix once and for all:

- a finite reduced descriptor digraph  $G = (V, E)$  together with a fixed labeling map  $e \mapsto T_e$  into nonnegative  $d \times d$  templates,
- the *support patterns* of all templates  $T_e$  (which entries are structurally allowed to be positive),
- and the perturbation metrics used in the robust certificate contract (entrywise for templates and a norm on costs).

A raw parameter  $\theta$  consists of:

- **template entry weights:** for each structurally allowed entry  $(i, j)$  of each label  $T_e$ , a positive real value  $a_{e,ij}$ ,
- **edge costs:** a real cost vector  $c_\theta : E \rightarrow \mathbb{R}$  (interpreted as reduced costs after the canonical normalization of Section 3).

To eliminate noncompact scaling degrees of freedom, we use the entrywise-max normalization: for each label  $e$ , rescale the nonzero entries of  $T_e(\theta)$  so that

$$\max_{i,j} (T_e(\theta))_{ij} = 1.$$

This normalization is continuous on the strictly positive locus and is compatible with the projective nature of the contraction conclusions.

## 5.2 The canonical compact family

Fix constants

$$0 < m_0 \leq 1, \quad C_0 > 0.$$

Define  $\Theta_{\text{can}} = \Theta_{\text{can}}(m_0, C_0)$  to be the set of all normalized parameters  $\theta$  such that:

- every structurally allowed entry of every template satisfies

$$(T_e(\theta))_{ij} \in [m_0, 1],$$

- the costs satisfy a mean-zero normalization and a uniform box bound

$$\sum_{e \in E} c_\theta(e) = 0, \quad |c_\theta(e)| \leq C_0 \text{ for all } e \in E.$$

**Lemma 5.1** (Compactness of  $\Theta_{\text{can}}$ ).  *$\Theta_{\text{can}}$  is compact in the product topology induced by the norms fixed in the robust certificate contract.*

*Proof.* Each template-entry coordinate lies in the compact interval  $[m_0, 1]$  and the cost coordinates lie in  $[-C_0, C_0]$  with one closed linear constraint  $\sum c = 0$ . Finite products of compact sets are compact, and closed subsets of compact sets are compact.  $\square$

### 5.3 What “completion” means on $\Theta_{\text{can}}$

On  $\Theta_{\text{can}}$  the global decider of Theorem 35.3 is now a concrete, fully specified procedure: the input is the machine description of  $\Theta_{\text{can}}$  (box constraints plus normalization), and the output is a `GlobalUniversalityReport`.

The only remaining mathematical distinction is *which outcome occurs*:

- If  $\Theta_{\text{can}}$  avoids the degeneracy loci (Definition 34.3) chamberwise, then the decider must certify (Corollary 35.6) and yields *good-global contraction universality within the certified class on the compact normalized family  $\Theta_{\text{can}}$* .
- Otherwise, the decider returns a *boundary-global regime atlas* locating the obstruction/degeneracy neighborhood where uniform contraction cannot hold on all of  $\Theta_{\text{can}}$  without further restriction.

In both cases the output object is the report schema of `global_universality_report_schema.json`; in the certified case, it can be compressed into a `GlobalUniversalityCertificate` object.

### 5.4 Making the net-certification inputs explicit (no hidden regularity)

The “global-on- $\Theta$  by finite net” route in Proposition 34.14 requires conservative continuity moduli. To prevent any impression that these are being assumed implicitly, the parameter-family descriptor schema includes a dedicated contract block `net_certification_contract`.

Concretely, when one wants a certified uniform margin lower bound via Lemma 34.13, the descriptor must declare:

- the parameter norm used to build an  $h$ -net;
- a conservative Lipschitz constant  $L_c$  for  $\theta \mapsto c(\theta)$  in  $\|\cdot\|_\infty$ ;
- a conservative Lipschitz constant  $L_A$  for  $\theta \mapsto A^{(k)}(\theta)$  in entrywise  $\|\cdot\|_{\infty, \text{ent}}$ ;
- a uniform entry bound  $M \geq \sup_{\theta \in \Theta} \max_{k,i,j} A_{ij}^{(k)}(\theta)$ .

On a frozen witness chamber, these inputs yield explicit envelopes for  $L_\gamma$  and  $L_\delta$  via Lemmas 34.18 and 34.19. The canonical file `theta_family_example.json` supplies this block for  $\Theta_{\text{can}}$  in a coordinate chart where the normalized variables are the parameters.

**Remark 5.2** (Concrete machine description files). *The repository includes a canonical region descriptor file `theta_family_example.json` matching the schema `theta_family_schema.json`. The example global report/certificate files `global_universality_report_ThetaCan_certified.json` and `global_universality_certificate_ThetaCan.json` illustrate the exact output structure on this target family.*

## 6 Aubry–Tight Core and the Induced Boundary Subshift

This section pins down the precise meaning of *core* used throughout the paper. The definition is finite, canonical, and built from the normalized reduced costs. It isolates the locally optimal transitions and the induced symbolic system that supports calibrated minimizers.

## 6.1 Calibrated subactions and local optimality

Fix the reduced graph  $G = (V, E)$  with costs  $c : E \rightarrow \mathbb{R}$  and the extremal mean  $\lambda_*$ .

**Definition 6.1** (Calibrated subaction). *A function  $u : V \rightarrow \mathbb{R}$  is a calibrated subaction (for  $c$  at level  $\lambda_*$ ) if for every edge  $e \in E$ ,*

$$u(\text{head}(e)) \leq u(\text{tail}(e)) + c(e) - \lambda_*. \quad (1)$$

Equivalently, the reduced cost

$$c_u(e) := c(e) - \lambda_* + u(\text{tail}(e)) - u(\text{head}(e)) \quad (2)$$

satisfies  $c_u(e) \geq 0$  for all  $e \in E$ .

**Remark 6.2.** *The potential  $\phi$  produced by Theorem 3.1 is a calibrated subaction in the sense of Definition 6.1, with  $c_\phi = c_u$ . Conversely, any calibrated subaction yields a valid canonical normalization.*

**Lemma 6.3** (Tight edges are exactly locally optimal edges). *Let  $u$  be any calibrated subaction. Then an edge  $e \in E$  is tight (i.e.  $c_u(e) = 0$ ) if and only if equality holds in (1):*

$$u(\text{head}(e)) = u(\text{tail}(e)) + c(e) - \lambda_*.$$

*In particular, the tight edge set  $E^{(0)} = \{e : c_u(e) = 0\}$  is exactly the set of one-step transitions that are locally optimal with respect to the calibrated subaction.*

*Proof.* By definition,  $c_u(e) = c(e) - \lambda_* + u(\text{tail}(e)) - u(\text{head}(e))$ . The inequality (1) is equivalent to  $c_u(e) \geq 0$ . Thus  $c_u(e) = 0$  holds exactly when (1) holds with equality.  $\square$

## 6.2 Aubry–tight core

Fix a calibrated subaction  $u$  (for example  $u = \phi$ ). Let  $G^{(0)} = (V^{(0)}, E^{(0)})$  denote the tight subgraph, i.e. the subgraph of locally optimal edges.

**Definition 6.4** (Aubry–tight core). *The Aubry–tight core (or simply the core) is the union of the strongly connected components of  $G^{(0)}$  that contain at least one directed cycle. Equivalently, it is the set of vertices that lie on some tight directed cycle. We denote this vertex set by  $\mathcal{A} \subseteq V^{(0)}$  and write  $G^{\mathcal{A}}$  for the induced subgraph of  $G^{(0)}$  on  $\mathcal{A}$ .*

**Remark 6.5.** *Throughout the paper, when we refer to a tight core, core SCC, or aperiodic core, we mean an SCC of  $G^{\mathcal{A}}$  as in Definition 6.4. No statement later relies on any stronger “eventual confinement” property for arbitrary mean-minimizers.*

## 6.3 Induced boundary subshift

The tight core defines a finite-type symbolic system capturing calibrated two-sided minimizers.

**Definition 6.6** (Boundary subshift induced by the tight core). *Let  $\Sigma_{\mathcal{A}} \subset (E^{(0)})^{\mathbb{Z}}$  be the set of bi-infinite edge sequences  $\omega = (\dots, e_{-1}, e_0, e_1, \dots)$  such that consecutive edges match and every vertex visited lies in  $\mathcal{A}$ . Equivalently,  $\Sigma_{\mathcal{A}}$  is the two-sided edge shift of the finite directed graph  $G^{\mathcal{A}}$ . See, e.g., Lind and Marcus [18] for standard background on shifts of finite type and edge shifts. We call  $(\Sigma_{\mathcal{A}}, \sigma)$  the boundary subshift induced by the Aubry–tight core.*

**Lemma 6.7** (Tight paths have extremal mean). *Let  $(e_0, \dots, e_{n-1})$  be a length- $n$  path consisting only of tight edges in  $E^{(0)}$ . Then*

$$\frac{1}{n} \sum_{t=0}^{n-1} c(e_t) = \lambda_* + \frac{u(v_n) - u(v_0)}{n},$$

where  $v_0$  is the initial vertex,  $v_n$  is the terminal vertex, and  $u$  is any calibrated subaction. In particular, along any one-sided infinite tight path, the asymptotic average cost equals  $\lambda_*$ .

*Proof.* For each tight edge  $e_t : v_t \rightarrow v_{t+1}$  we have  $c_u(e_t) = 0$ , i.e.  $c(e_t) - \lambda_* + u(v_t) - u(v_{t+1}) = 0$ . Summing from  $t = 0$  to  $n - 1$  yields  $\sum_{t=0}^{n-1} c(e_t) = n\lambda_* + u(v_n) - u(v_0)$ . Divide by  $n$ .  $\square$

**Remark 6.8.** *The boundary subshift  $\Sigma_{\mathcal{A}}$  is the natural symbolic carrier of calibrated (two-sided) minimizers. The gap-free boundary classification later in the paper is performed SCC-by-SCC on this core subshift. When working with a specific one-sided minimizing trajectory, the relevant input is the set of tight times (Proposition 3.4) and the positive-density core selection (Proposition 3.5); no stronger trapping statement is assumed.*

## 7 Computing the extremal mean lambda-star, a calibrating potential, and the tight subgraph

This appendix records an explicit finite procedure to compute the objects in Section 3. All steps are explicit finite graph algorithms; we include them to make the normalization fully machine-checkable.

### 7.1 Computing the minimal cycle mean

Let  $G = (V, E)$  with  $|V| = n$ ,  $|E| = m$ , and costs  $c : E \rightarrow \mathbb{R}$ . The minimal cycle mean

$$\lambda_* = \min_{\gamma \text{ cycle}} \frac{1}{|\gamma|} \sum_{e \in \gamma} c(e)$$

can be computed in  $O(nm)$  time using Karp's algorithm.

---

**Algorithm 2** Karp minimal cycle mean

---

**Require:** Directed graph  $G = (V, E)$ , costs  $c(e)$

**Ensure:**  $\lambda_*$

- 1: Choose any ordering of vertices and initialize  $d_0(v) = 0$  for all  $v \in V$
  - 2: **for**  $k = 1$  to  $n$  **do**
  - 3:   **for all**  $v \in V$  **do**
  - 4:      $d_k(v) \leftarrow \min_{(u \rightarrow v) \in E} (d_{k-1}(u) + c(u \rightarrow v))$
  - 5:   **end for**
  - 6: **end for**
  - 7: **for all**  $v \in V$  **do**
  - 8:    $M(v) \leftarrow \max_{0 \leq k \leq n-1} \frac{d_n(v) - d_k(v)}{n - k}$
  - 9: **end for**
  - 10:  $\lambda_* \leftarrow \min_{v \in V} M(v)$
  - 11: **return**  $\lambda_*$
-

## 7.2 Computing a calibrating potential

Set  $\tilde{c}(e) = c(e) - \lambda_*$ . Then  $G$  has no negative cycles with respect to  $\tilde{c}$  by definition of  $\lambda_*$ . A potential  $\phi$  such that  $c_\phi(e) = \tilde{c}(e) + \phi(u) - \phi(v) \geq 0$  for all  $e : u \rightarrow v$  can be obtained from shortest-path distances.

---

### Algorithm 3 Calibrating potential via shortest paths

---

**Require:** Graph  $G = (V, E)$ , reduced costs  $\tilde{c}(e) = c(e) - \lambda_*$

**Ensure:** Potential  $\phi$  with  $c_\phi(e) \geq 0$  for all  $e$

- 1: Add a super-source  $s$  with zero-cost edges  $(s \rightarrow v)$  to every  $v \in V$
  - 2: Run Bellman–Ford on  $(V \cup \{s\}, E \cup \{(s \rightarrow v)\})$  with costs  $\tilde{c}$  to compute distances  $\phi(v) = \text{dist}(s, v)$
  - 3: **return**  $\phi$
- 

**Remark 7.1.** *Because there are no negative cycles, Bellman–Ford terminates and returns finite distances. The inequality  $\phi(v) \leq \phi(u) + \tilde{c}(u \rightarrow v)$  is exactly  $c_\phi(u \rightarrow v) \geq 0$ .*

## 7.3 Extracting the tight subgraph and its SCC structure

Given  $\phi$ , compute  $c_\phi(e)$  and define  $E^{(0)} = \{e : c_\phi(e) = 0\}$  as in Definition 3.2. This yields the tight subgraph  $G^{(0)}$ . SCC decomposition runs in  $O(n + m)$  time (Tarjan/Kosaraju). Aperiodicity of an SCC is certified by computing the gcd of its cycle lengths (linear-time in the SCC via a spanning tree + back-edge depth differences).

---

### Algorithm 4 Tight subgraph + SCC periodicity labels

---

**Require:**  $G = (V, E)$ , costs  $c$ ,  $\lambda_*$ , potential  $\phi$

**Ensure:** Tight graph  $G^{(0)}$  and SCCs labeled periodic/aperiodic

- 1:  $E^{(0)} \leftarrow \{u \rightarrow v \in E : c(u \rightarrow v) - \lambda_* + \phi(u) - \phi(v) = 0\}$
  - 2: Compute SCC decomposition of  $G^{(0)}$
  - 3: **for all** SCCs  $H$  **do**
  - 4:   Compute  $\text{per}(H) = \text{gcd}$  of cycle lengths in  $H$
  - 5:   **if**  $\text{per}(H) = 1$  and  $|H| > 1$  **then** label aperiodic
  - 6:   **else** label periodic
  - 7:   **end if**
  - 8: **end for**
- 

## 7.4 Complexity summary

All objects in Section 3 are computable in polynomial time from the reduced finite graph:

- Karp minimal mean:  $O(nm)$ .
- Bellman–Ford potential:  $O(nm)$ .
- Tight graph extraction:  $O(m)$ .
- SCC decomposition:  $O(n + m)$ .

Thus the canonical normalization and tight geometry can be produced and checked algorithmically as part of the finite certification pipeline.

## 8 Calibrated Minimizers and Selection Principles

This section isolates the precise sense in which one may *restrict* attention to calibrated (locally optimal) minimizers, while keeping clear what this does *not* imply for arbitrary one-sided mean-minimizers. The main point is that calibration is an *edgewise* (one-step) equality condition, so calibrated minimizers live entirely on the tight subgraph, whereas general minimizers are only asymptotically tight.

### 8.1 Calibrated minimizers

Fix a calibrated subaction  $u : V \rightarrow \mathbb{R}$  in the sense of Definition 6.1, and write  $c_u$  for the associated reduced costs (2).

**Definition 8.1** (Calibrated bi-infinite minimizer). *A bi-infinite admissible edge sequence  $\omega = (\dots, e_{-1}, e_0, e_1, \dots) \in E^{\mathbb{Z}}$  is a calibrated minimizer for  $u$  if consecutive edges match and*

$$c_u(e_t) = 0 \quad \text{for all } t \in \mathbb{Z}. \quad (3)$$

*Equivalently,  $\omega$  is a bi-infinite path in the tight subgraph  $G^{(0)} = (V^{(0)}, E^{(0)})$  defined by  $u$ .*

**Lemma 8.2** (Calibrated minimizers are supported on tight edges). *If  $\omega$  is a calibrated minimizer for  $u$ , then every edge of  $\omega$  lies in  $E^{(0)} = \{e \in E : c_u(e) = 0\}$ . In particular, every calibrated minimizer is minimizing with asymptotic mean cost  $\lambda_*$ .*

*Proof.* The first claim is immediate from (3). For the mean cost, sum the identity  $c(e_t) - \lambda_* + u(v_t) - u(v_{t+1}) = 0$  along any finite window, exactly as in Lemma 6.7, and divide by the window length.  $\square$

### 8.2 Existence on the Aubry–tight core

The Aubry–tight core  $\mathcal{A}$  from Definition 6.4 is designed so that tight cycles exist through every vertex of  $\mathcal{A}$ . This yields calibrated minimizers without any recurrence hypothesis.

**Proposition 8.3** (Periodic calibrated minimizers on the core). *For every vertex  $v \in \mathcal{A}$  there exists a periodic bi-infinite calibrated minimizer  $\omega \in \Sigma_{\mathcal{A}}$  that visits  $v$ . Consequently,  $\Sigma_{\mathcal{A}}$  is nonempty whenever  $\mathcal{A} \neq \emptyset$ .*

*Proof.* By Definition 6.4, the vertex  $v$  lies on a directed tight cycle  $\gamma$  in  $G^{(0)}$ . Let  $w$  be the finite edge word given by  $\gamma$  starting and ending at  $v$ . Repeating  $w$  periodically in both time directions produces a bi-infinite admissible edge sequence  $\omega \in (E^{(0)})^{\mathbb{Z}}$ . Then  $c_u(e_t) = 0$  for all  $t$ , so  $\omega$  is calibrated by Definition 8.1.  $\square$

### 8.3 Minimizing measures are tight

The previous subsection constructs explicit calibrated minimizers on  $\mathcal{A}$ . For applications, it is also useful to know that every *invariant* minimizer is supported on the tight shift.

**Proposition 8.4** (Invariant minimizers live on the tight shift). *Let  $\mu$  be a  $\sigma$ -invariant probability measure on the two-sided edge shift of  $G$ . If  $\mu$  minimizes the average cost, i.e.*

$$\int c d\mu = \lambda_*,$$

*then  $\mu$  is supported on the tight edge shift  $(E^{(0)})^{\mathbb{Z}}$ . In particular,  $c_u(e) = 0$  for  $\mu$ -almost every edge occurrence.*

*Proof.* By Definition 6.1, the reduced cost  $c_u(e) \geq 0$  for all edges. For a bi-infinite path  $\omega$  with vertex sequence  $(v_t)_{t \in \mathbb{Z}}$  we have

$$c_u(e_t) = c(e_t) - \lambda_* + u(v_t) - u(v_{t+1}).$$

Integrating with respect to a  $\sigma$ -invariant measure  $\mu$  yields

$$\int c_u d\mu = \int (c - \lambda_*) d\mu + \int (u \circ \text{tail} - u \circ \text{head}) d\mu.$$

The last integral vanishes by  $\sigma$ -invariance (it is the integral of a coboundary), hence

$$\int c_u d\mu = \int c d\mu - \lambda_*.$$

If  $\int c d\mu = \lambda_*$ , then  $\int c_u d\mu = 0$ . Since  $c_u \geq 0$ , it follows that  $c_u = 0$   $\mu$ -almost surely. Equivalently,  $\mu$  is supported on the set of paths using only edges in  $E^{(0)}$ .  $\square$

**Corollary 8.5** (Calibrated representatives exist in minimizing supports). *If  $\mu$  is a  $\sigma$ -invariant minimizing measure, then  $\mu$ -almost every  $\omega$  is a calibrated minimizer for  $u$ . In particular, the support of any minimizing measure is contained in the boundary subshift  $\Sigma_{\mathcal{A}}$ .*

*Proof.* The first statement is immediate from Proposition 8.4. For the second, note that any bi-infinite tight path must lie in an SCC of  $G^{(0)}$  that contains a cycle. By Definition 6.4, the union of such SCCs is exactly the Aubry–tight core  $\mathcal{A}$ , so every bi-infinite tight path lies in  $\Sigma_{\mathcal{A}}$ .  $\square$

## 8.4 What calibration does and does not buy for one-sided minimizers

The results above justify a common proof strategy: when proving statements about the *symbolic carrier* of extremality (the boundary subshift and its SCCs), one may work entirely on calibrated minimizers. However, one must not confuse this with a statement about a *given* one-sided minimizing trajectory.

**Proposition 8.6** (Calibrated selection from a one-sided minimizer). *Let  $\pi = (e_0, e_1, \dots)$  be a one-sided minimizing path. Then there exists a bi-infinite calibrated minimizer  $\omega$  and a sequence of times  $n_k \rightarrow \infty$  such that  $\sigma^{n_k} \pi \rightarrow \omega$  in the product topology on  $E^{\mathbb{Z}}$ . Equivalently, the  $\omega$ -limit set of  $\pi$  contains a calibrated minimizer.*

*Proof.* By Proposition 3.4, the set  $T = \{t \geq 0 : e_t \in E^{(0)}\}$  has asymptotic density 1. Fix an integer  $m \geq 1$ . Because  $T$  has density 1, there exist arbitrarily large times  $n$  such that the window  $\{n - m, \dots, n + m\}$  is contained in  $T$ . Choose  $n(m)$  with this property. The shift space  $E^{\mathbb{Z}}$  is compact, so by a diagonal subsequence argument there exists a sequence  $n_k \rightarrow \infty$  with  $\sigma^{n_k} \pi$  converging to some bi-infinite admissible limit path  $\omega$ . By construction of the windows, for each fixed coordinate  $t \in \mathbb{Z}$ , the edge at position  $t$  in  $\sigma^{n_k} \pi$  belongs to  $E^{(0)}$  for all sufficiently large  $k$ ; hence the limiting edge  $e_t(\omega)$  also lies in  $E^{(0)}$ . Therefore  $\omega$  uses only tight edges and is a calibrated minimizer by Definition 8.1.  $\square$

**Remark 8.7** (Scope and limitations). *Proposition 8.6 is a selection statement: from a minimizing trajectory one can extract calibrated minimizers as limit points. It does not imply that the original trajectory is eventually tight, trapped in a single tight SCC, or recurrent. The only trajectory-level facts used later are*

- density-1 tightness (Proposition 3.4), and

- *positive-lower-density selection of at least one tight SCC (Proposition 3.5).*

Any stronger recurrence input (bounded gaps of a certified Doeblin word, or a syndetic modulus from an automaton) is treated as a separate layer.

## 9 Minimality Route: what can and cannot be forced on the tight core

This section clarifies the “minimality” upgrade route for the minimizing dynamics on the tight core. The conclusion is sharp and finite-witness: because the boundary carrier  $\Sigma_{\mathcal{A}}$  is a shift of finite type, true topological minimality (and hence linear recurrence) occurs only in the purely periodic case. Accordingly, minimality is not an automatic consequence of calibration or tightness; the correct replacement for gap-control is the *finite* recurrence certificate layer (RC1) and the slack-forcing route developed later.

### 9.1 Minimal SFTs are periodic

We record the elementary structural fact that blocks the naive minimality upgrade on SFT carriers.

**Definition 9.1** (Minimality). *A two-sided subshift  $(X, \sigma)$  is minimal if every orbit is dense in  $X$ . Equivalently,  $X$  has no proper nonempty closed  $\sigma$ -invariant subsets.*

**Lemma 9.2** (A minimal edge shift is a single cycle). *Let  $H$  be a finite directed graph and let  $X_H$  be its two-sided edge shift. If  $(X_H, \sigma)$  is minimal, then  $X_H$  consists of a single periodic orbit. Equivalently,  $H$  is a directed cycle (every vertex has in-degree and out-degree equal to 1 in the active component).*

*Proof.* If  $X_H$  is nonempty, then  $H$  contains a directed cycle, hence  $X_H$  contains a periodic point  $\omega$  obtained by repeating that cycle. The orbit closure of a periodic point is finite. If  $X_H$  were minimal, the orbit closure of  $\omega$  would have to equal all of  $X_H$ . Therefore  $X_H$  is finite. An edge shift over a finite graph is finite if and only if the underlying active component is a directed cycle, because any branching (a vertex with two distinct outgoing edges or two distinct incoming edges) produces at least two distinct infinite paths. Thus  $H$  must be a directed cycle and  $X_H$  is a single periodic orbit.  $\square$

### 9.2 Implication for the minimizing carrier

Recall that the boundary carrier  $\Sigma_{\mathcal{A}}$  defined in Definition 6.6 is the two-sided edge shift of the finite graph  $G^{\mathcal{A}}$ . Hence it is a shift of finite type.

**Proposition 9.3** (Minimality on  $\Sigma_{\mathcal{A}}$  is a periodic obstruction). *If the boundary carrier  $(\Sigma_{\mathcal{A}}, \sigma)$  is minimal, then every SCC of  $G^{\mathcal{A}}$  is a directed cycle. In particular, the only minimal regime for the tight-core carrier is the purely periodic regime.*

*Conversely, if a tight SCC  $H \subseteq G^{\mathcal{A}}$  is a directed cycle, then its edge shift is minimal and every tight trajectory supported in  $H$  is periodic.*

*Proof.* Apply Lemma 9.2 to each nonempty irreducible component. If  $\Sigma_{\mathcal{A}}$  is minimal and nonempty, it cannot decompose into multiple invariant components, so it coincides with the shift of a single SCC. That SCC must be a directed cycle, and the conclusions follow. The converse direction is immediate for a directed cycle.  $\square$

### 9.3 The exact missing property and the correct replacement

The previous proposition isolates the precise missing property: without an explicit *finite* obstruction witness forcing the tight SCC to be a directed cycle, minimality (and linear recurrence) cannot be deduced from calibration alone. In particular, tightness is compatible with:

- multiple tight cycles in the same tight SCC,
- branching tight choices (out-degree  $\geq 2$  along tight edges),
- aperiodic tight SCCs that are topologically transitive but far from minimal.

These shapes are not pathologies; they occur naturally and are consistent with the obstruction taxonomy OG1–OG3. Therefore, the minimality route is not the right vehicle for gap-free recurrence on minimizers.

**Remark 9.4** (Finite recurrence certificate as the correct “minimality substitute”). *What is actually used in the contraction upgrade is not minimality of the entire tight SCC, but a finite return-control statement for a specific certified Doeblin word  $w$ . This is exactly the role of the recurrence certificate layer (RC1): for a fixed tight SCC  $H$  and a candidate word  $w$  supported in  $H$ , one either*

- *produces an explicit bounded-gap modulus  $L_{\text{syn}}$  guaranteeing syndetic returns of  $w$  along every infinite tight path in  $H$ , or*
- *returns a finite avoidance-cycle witness showing that some periodic tight path avoids  $w$  entirely.*

*This finite dichotomy is the sharp replacement for minimality in the gap-free regime.*

## 10 Slack-Forcing from Doeblin-Word Avoidance

This section isolates a quantitative principle that will be used later in the forcing dichotomy: once a Doeblin word is certified on a tight aperiodic SCC and the bounded-gap recurrence certificate (RC1) holds for tight trajectories, then any attempt to avoid that word over long time windows necessarily incurs a *linear* amount of reduced-cost slack.

### 10.1 Setup and constants

Fix the calibrated potential  $\phi$  and reduced costs  $c_\phi(e) \geq 0$  as in (2). Recall the tight edge set  $E^{(0)} = \{e : c_\phi(e) = 0\}$  and the tight graph  $G^{(0)} = (V^{(0)}, E^{(0)})$ .

Let

$$\alpha := \min\{c_\phi(e) : e \in E \setminus E^{(0)}\}, \quad (4)$$

with the convention  $\alpha = +\infty$  if  $E^{(0)} = E$ . When  $E^{(0)} \neq E$ , we have  $\alpha > 0$  by finiteness of  $E$ . This constant already appears in Proposition 3.4.

**Definition 10.1** (Slack count and slack mass on a window). *For a finite edge-word  $\gamma = (e_0, \dots, e_{L-1})$ , define*

$$\text{SlackCount}(\gamma) := |\{0 \leq t < L : e_t \notin E^{(0)}\}|, \quad \text{SlackMass}(\gamma) := \sum_{t=0}^{L-1} c_\phi(e_t).$$

By construction,  $\text{SlackMass}(\gamma) \geq \alpha \cdot \text{SlackCount}(\gamma)$  whenever  $\alpha < \infty$ .

## 10.2 RC1 turns word-avoidance into a bound on tight-run lengths

Let  $H$  be a tight SCC of  $G^{(0)}$  and let  $w$  be a finite tight edge-word supported in  $H$ . Assume that the recurrence certificate **(RC1)** holds for  $(H, w)$  in the sense of Definition 15.1. Let  $L_{\text{syn}}(H, w)$  denote the explicit syndetic return modulus from Lemma 15.6. Set

$$L_0 := L_{\text{syn}}(H, w) + |w|. \quad (5)$$

**Lemma 10.2** (No long tight runs in  $H$  can avoid  $w$ ). *Assume **(RC1)** holds for  $(H, w)$  and let  $L_0$  be as in (5). If  $\gamma = (e_0, \dots, e_{L-1})$  is a finite path that stays inside  $H$  and uses only tight edges  $e_t \in E^{(0)}$ , then  $\gamma$  contains an occurrence of  $w$  provided  $L \geq L_0$ . Equivalently, any tight path segment inside  $H$  with no occurrence of  $w$  has length at most  $L_0 - 1$ .*

*Proof.* Because  $H$  is strongly connected and contains a directed cycle, any finite tight path segment inside  $H$  can be extended forward to an infinite tight path in  $H$ . Apply Lemma 15.6 to that infinite tight extension: it contains occurrences of  $w$  with gaps between *starting indices* bounded by  $L_{\text{syn}}(H, w)$ . In particular, there exists an occurrence of  $w$  whose start index is at most  $L_{\text{syn}}(H, w)$ . Since an occurrence of  $w$  occupies  $|w|$  consecutive edges, the entire occurrence lies within the first  $L_{\text{syn}}(H, w) + |w| = L_0$  edges. Thus any tight segment of length  $L \geq L_0$  must contain an occurrence of  $w$ .  $\square$

## 10.3 Linear slack forcing on word-avoiding windows

We now turn Lemma 10.2 into a quantitative lower bound on slack.

**Proposition 10.3** (Avoiding  $w$  for length  $L$  forces linear slack). *Let  $H$  be a tight SCC of  $G^{(0)}$  and let  $w$  be a tight word supported in  $H$  such that **(RC1)** holds for  $(H, w)$ . Let  $L_0$  be as in (5). Consider any finite path segment  $\gamma = (e_0, \dots, e_{L-1})$  that stays inside  $H$  and contains no occurrence of  $w$ . Then*

$$\text{SlackCount}(\gamma) \geq \max\left\{0, \left\lceil \frac{L - L_0}{L_0 + 1} \right\rceil\right\}, \quad (6)$$

and consequently, if  $\alpha < \infty$ ,

$$\text{SlackMass}(\gamma) \geq \alpha \cdot \max\left\{0, \left\lceil \frac{L - L_0}{L_0 + 1} \right\rceil\right\}. \quad (7)$$

In particular,  $\text{SlackMass}(\gamma)$  grows at least linearly in  $L$  once  $L > L_0$ .

*Proof.* If  $\gamma$  contains no tight edges, then  $\text{SlackCount}(\gamma) = L$  and (6) holds. Assume  $\gamma$  contains at least one tight edge. Decompose  $\gamma$  into maximal consecutive blocks of tight edges (tight runs) separated by slack edges. Let  $k$  be the number of tight runs and let  $s = \text{SlackCount}(\gamma)$  be the number of slack edges. Then necessarily  $k \leq s + 1$ .

By Lemma 10.2, every tight run inside  $H$  that avoids  $w$  has length at most  $L_0 - 1$ . Hence the total number of tight edges in  $\gamma$  is at most  $k(L_0 - 1)$ . Therefore

$$L = (\text{number of tight edges}) + s \leq k(L_0 - 1) + s \leq (s + 1)(L_0 - 1) + s = s(L_0 + 1) + (L_0 - 1).$$

Rearranging gives  $s \geq (L - L_0)/(L_0 + 1)$ . Since  $s$  is an integer and  $s \geq 0$ , we obtain (6). Finally, (7) follows from  $\text{SlackMass}(\gamma) \geq \alpha \text{SlackCount}(\gamma)$ .  $\square$

**Remark 10.4** (What this does and does not say). *Proposition 10.3 is a local forcing statement: it applies to windows that stay inside a fixed tight SCC  $H$  where **(RC1)** holds for a chosen word  $w$ . If **(RC1)** fails, Lemma 15.3 provides a tight periodic witness that avoids  $w$  with zero slack, showing that no positive lower bound of the form (7) can hold in that case. Excursions that leave  $H$  and later return are handled separately in Section 11.*

## 11 Return Cost for Excursions Out of a Tight SCC

This section isolates a second forcing mechanism that complements Section 10: even if a trajectory attempts to avoid a certified Doeblin word by *leaving* the relevant tight SCC, any later *return* to that SCC forces reduced-cost slack. This turns repeated excursions (a natural avoidance strategy) into a measurable slack burden.

### 11.1 A structural fact: tight SCCs do not admit tight returns from outside

Fix the calibrated potential  $\phi$  and reduced costs  $c_\phi$ . Let  $G^{(0)}$  be the tight graph. Let  $H$  be an SCC of  $G^{(0)}$ .

**Definition 11.1** (Excursion and return). *Let  $\pi = (e_0, e_1, \dots)$  be a one-sided path in the full graph  $G$  with vertex sequence  $(v_t)_{t \geq 0}$ , where  $e_t : v_t \rightarrow v_{t+1}$ . A time  $t \geq 1$  is called a return time to  $H$  if  $v_t \in H$  and  $v_{t-1} \notin H$ . An excursion is a maximal time interval during which  $v_t \notin H$  following a departure from  $H$  and ending at the next return time (if it exists).*

**Lemma 11.2** (Returning to a tight SCC forces slack). *Let  $\gamma = (e_0, \dots, e_{L-1})$  be a finite path segment in  $G$  with vertex sequence  $(v_0, \dots, v_L)$ . Assume  $v_0 \in H$  and  $v_L \in H$ , and assume that  $v_t \notin H$  for at least one intermediate time  $t$  with  $0 < t < L$ . Then  $\gamma$  contains at least one non-tight edge, i.e.  $\text{SlackCount}(\gamma) \geq 1$ . In particular, if  $\alpha < \infty$  (as in (4)), then  $\text{SlackMass}(\gamma) \geq \alpha$ .*

*Proof.* Suppose for contradiction that every edge of  $\gamma$  is tight, so  $\gamma$  is a path in the tight graph  $G^{(0)}$ . Then in  $G^{(0)}$  there exists a tight path from  $H$  to a vertex outside  $H$  (since some  $v_t \notin H$  occurs) and a tight path back from outside to  $H$  (since  $v_L \in H$ ). This implies that in the SCC condensation DAG of  $G^{(0)}$  there is a directed path from the SCC  $H$  to a different SCC and a directed path back to  $H$ , creating a directed cycle in the condensation. But the SCC condensation is a DAG, so such a cycle is impossible. Therefore  $\gamma$  must contain a non-tight edge, proving  $\text{SlackCount}(\gamma) \geq 1$ . The bound on  $\text{SlackMass}$  follows from  $\text{SlackMass}(\gamma) \geq \alpha \text{SlackCount}(\gamma)$ .  $\square$

### 11.2 Quantifying slack per return

**Proposition 11.3** (Slack lower bound per number of returns). *Let  $\pi = (e_0, e_1, \dots)$  be any one-sided path in  $G$  and let  $R_n(H)$  be the number of return times to  $H$  among  $\{1, \dots, n\}$  (Definition 11.1). Then for every  $n \geq 1$ ,*

$$R_n(H) \leq \sum_{t=0}^{n-1} \mathbf{1}_{\{e_t \notin E^{(0)}\}}, \quad (8)$$

and consequently, if  $\alpha < \infty$ ,

$$\sum_{t=0}^{n-1} c_\phi(e_t) \geq \alpha \cdot R_n(H). \quad (9)$$

*Proof.* Each return time to  $H$  completes an excursion that starts in  $H$ , leaves  $H$ , and ends in  $H$ . Apply Lemma 11.2 to the finite path segment spanning that excursion. Different excursions are edge-disjoint in time, so each return time accounts for at least one distinct non-tight edge. This yields (8), and (9) follows from (4).  $\square$

### 11.3 Connection to word-avoidance strategies

Let  $H$  be a tight SCC and let  $w$  be a tight word supported in  $H$ . If **(RC1)** holds for  $(H, w)$ , then Section 10 shows that long word-avoidance windows *inside*  $H$  force linear slack. Proposition 11.3 shows that word-avoidance by repeated *excursions* out of  $H$  also forces slack.

**Remark 11.4** (Excursions as avoidance). *Assume **(RC1)** holds for  $(H, w)$  and let  $L_0$  be as in (5). If a path visits  $H$  frequently but attempts to avoid  $w$ , then it must do at least one of the following on most long windows:*

- *introduce slack edges while staying in  $H$ , which is quantitatively penalized by Proposition 10.3;*
- *leave  $H$  before a tight run of length  $L_0$  can occur and later return, which is penalized by Proposition 11.3.*

*This is the mechanism used in the forcing dichotomy that follows: either  $w$  recurs with bounded gaps on the relevant tight core, or slack density is bounded below by a positive constant (contradicting asymptotic tightness for mean-minimizers).*

## 12 Forcing Dichotomy: Doeblin Frequency versus Slack Density

This section upgrades the qualitative “tight-density” information available for general minimizers into a quantitative forcing statement. It combines:

- the **RC1** avoidance certificate for a word  $w$  inside a fixed tight aperiodic SCC  $H$  (Definition 15.1), and
- the **return-cost** principle for excursions out of  $H$  (Proposition 11.3).

The output is a dichotomy that is strong enough to feed the Doeblin $\Rightarrow$ Hilbert-contraction chain: either the Doeblin word occurs with *uniform positive lower density* along the trajectory, or the reduced-cost slack has *positive lower density* (hence the trajectory cannot be minimizing).

### 12.1 Setup

Fix the calibrated potential  $\phi$  and reduced costs  $c_\phi(e) \geq 0$  as in (2). Let  $E^{(0)} = \{e \in E : c_\phi(e) = 0\}$  be the tight edge set and let  $G^{(0)} = (V^{(0)}, E^{(0)})$  be the tight graph.

Fix a tight SCC  $H$  of  $G^{(0)}$  and a finite edge-word  $w$  supported in  $H$  such that **(RC1)** holds for  $(H, w)$ . Let  $L_{\text{syn}}(H, w)$  be the explicit syndetic modulus from Lemma 15.6 and set

$$L_0 := L_{\text{syn}}(H, w) + |w| \quad (10)$$

as in (5). Let

$$\alpha := \min\{c_\phi(e) : e \in E \setminus E^{(0)}\} \in (0, \infty] \quad (11)$$

as in (4).

For a one-sided path  $\pi = (e_0, e_1, \dots)$  in  $G$  define the length- $n$  slack mass

$$\text{SlackMass}_n(\pi) := \sum_{t=0}^{n-1} c_\phi(e_t). \quad (12)$$

Define the tight time count in  $H$  by

$$T_H(n; \pi) := \sum_{t=0}^{n-1} \mathbf{1}_{\{e_t \in E^{(0)} \text{ and } e_t \text{ is supported in } H\}}, \quad (13)$$

and let  $R_n(H)$  be the number of return times to  $H$  among  $\{1, \dots, n\}$  as in Definition 11.1.

Finally, let  $N_w(n; \pi)$  denote the number of starting indices  $0 \leq i \leq n-1$  at which the word  $w$  occurs along  $\pi$  (i.e.  $e_i e_{i+1} \cdots e_{i+|w|-1} = w$ ).

## 12.2 A counting lemma on maximal tight runs

**Definition 12.1** (Maximal tight-in- $H$  runs). *For fixed  $n$ , consider the set of indices  $t \in \{0, \dots, n-1\}$  for which  $e_t$  is tight and supported in  $H$ . The maximal contiguous intervals of such indices determine disjoint finite subpaths*

$$\gamma^{(1)}, \dots, \gamma^{(m)}$$

*which we call the maximal tight-in- $H$  runs of  $\pi$  up to time  $n$ . Write  $\ell_j := |\gamma^{(j)}|$  for the length (number of edges) of the  $j$ -th run. Then  $\sum_{j=1}^m \ell_j = T_H(n; \pi)$  and  $m-1 \leq R_n(H)$ .*

**Lemma 12.2** (Each long tight run forces many occurrences of  $w$ ). *Assume **(RC1)** holds for  $(H, w)$  and let  $L_0$  be as in (10). Let  $\gamma$  be a finite tight-in- $H$  run of length  $\ell$ . Then  $\gamma$  contains at least  $\lfloor \ell/L_0 \rfloor$  occurrences of  $w$ .*

*Proof.* Partition  $\gamma$  into consecutive disjoint blocks of length  $L_0$ , discarding at most a final remainder block of length  $< L_0$ . Each full block is a tight path in  $H$  of length  $L_0$ . By Lemma 10.2, no such block can avoid  $w$ , so each full block contains an occurrence of  $w$  supported entirely inside that block. Disjoint blocks cannot share an occurrence, hence the number of occurrences is at least the number of full blocks, which is  $\lfloor \ell/L_0 \rfloor$ .  $\square$

**Lemma 12.3** (Occurrences versus returns). *For every  $n \geq 1$  and every path  $\pi$ ,*

$$N_w(n; \pi) \geq \frac{T_H(n; \pi)}{L_0} - R_n(H) - 1. \quad (14)$$

*Proof.* Let  $\gamma^{(1)}, \dots, \gamma^{(m)}$  be the maximal tight-in- $H$  runs up to time  $n$  (Definition 12.1) with lengths  $\ell_1, \dots, \ell_m$ . By Lemma 12.2,

$$N_w(n; \pi) \geq \sum_{j=1}^m \left\lfloor \frac{\ell_j}{L_0} \right\rfloor \geq \sum_{j=1}^m \left( \frac{\ell_j}{L_0} - 1 \right) = \frac{T_H(n; \pi)}{L_0} - m.$$

Since  $m \leq R_n(H) + 1$ , we obtain (14).  $\square$

## 12.3 The forcing dichotomy

**Proposition 12.4** (Doebelin frequency vs slack density). *Assume **(RC1)** holds for  $(H, w)$  and let  $L_0$  and  $\alpha$  be as in (10)–(11). Then for every  $n \geq 1$  and every one-sided path  $\pi$  in  $G$ ,*

$$N_w(n; \pi) \geq \frac{T_H(n; \pi)}{L_0} - \frac{\text{SlackMass}_n(\pi)}{\alpha} - 1, \quad (15)$$

*with the convention that  $\text{SlackMass}_n(\pi)/\alpha = 0$  if  $\alpha = \infty$ . Equivalently,*

$$\text{SlackMass}_n(\pi) \geq \alpha \cdot \left( \frac{T_H(n; \pi)}{L_0} - N_w(n; \pi) - 1 \right)_+. \quad (16)$$

*Proof.* Combine Lemma 12.3 with the slack-per-return bound (9) from Proposition 11.3, which yields  $R_n(H) \leq \text{SlackMass}_n(\pi)/\alpha$  (for  $\alpha < \infty$ ). If  $\alpha = \infty$ , then  $E \setminus E^{(0)} = \emptyset$  and  $\text{SlackMass}_n(\pi) = 0$  identically, so (15) reduces to (14).  $\square$

**Corollary 12.5** (Forced positive density of the Doeblin word). *Assume (RC1) holds for  $(H, w)$  and let  $L_0$  be as in (10). Let  $\pi$  be any path such that*

$$\liminf_{n \rightarrow \infty} \frac{T_H(n; \pi)}{n} \geq \theta > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\text{SlackMass}_n(\pi)}{n} = 0.$$

*Then*

$$\liminf_{n \rightarrow \infty} \frac{N_w(n; \pi)}{n} \geq \frac{\theta}{L_0}.$$

*In particular,  $w$  occurs along  $\pi$  with a uniform positive lower density depending only on the forcing constants.*

*Proof.* Divide (15) by  $n$  and take  $\liminf_{n \rightarrow \infty}$ , using the hypotheses.  $\square$

**Remark 12.6** (Interpretation). *Corollary 12.5 is the promised forcing upgrade: in the regime where a minimizer has nontrivial tight mass in a fixed aperiodic tight SCC  $H$  and reduced-cost slack density 0, (RC1) forces a quantitative supply of Doeblin blocks along the trajectory. This is sufficient for explicit exponential projective contraction (see the rate pack in Section 20), without needing global bounded-gap recurrence.*

## 13 Descriptor Faces as Finite Certificates (Eliminating Interface Assumptions)

Two interface notions recur throughout the gap-free theory on a tight SCC  $H$ : *active-face invariance* and *support-connectivity on the active face*. In this section we turn both into purely finite, checkable certificates extracted directly from the reduced descriptor graph and template sparsity.

### 13.1 Canonical active face per descriptor state

In the descriptor reduction, each vertex  $v \in V$  corresponds to a finite descriptor encoding which coordinates are *potentially active* and which are structurally zero under admissible propagation. Accordingly, each descriptor vertex carries a canonical support set

$$S(v) \subseteq \{1, \dots, d\}.$$

**Definition 13.1** (Face map and canonical active set). *A face map for the reduced graph is a function  $S : V \rightarrow 2^{\{1, \dots, d\}}$  such that for each edge  $e : u \rightarrow v$  labeled by a template (or fixed-length block)  $T_e$  we have:*

- (i) (**Forward invariance**)  $T_e(\mathbb{R}_{>0}^{S(u)}) \subseteq \mathbb{R}_{>0}^{S(v)}$ ;
- (ii) (**No spurious activation**) If  $x \in \mathbb{R}_{>0}^{S(u)}$ , then  $(T_e x)_i = 0$  for all  $i \notin S(v)$ .

**Remark 13.2.** *Definition 13.1 is checkable from template sparsity: it amounts to verifying that the support pattern of each  $T_e$  maps the indicator of  $S(u)$  into  $S(v)$ .*

### 13.2 Face stability on SCCs as a finite fixed-point

**Lemma 13.3** (SCC face stability by intersection closure). *Let  $H$  be an SCC of the reduced graph and let  $S : V \rightarrow 2^{\{1, \dots, d\}}$  be a face map. Define the SCC face by*

$$S(H) := \bigcap_{v \in H} S(v).$$

*Then for every edge  $e : u \rightarrow v$  inside  $H$ ,*

$$T_e(\mathbb{R}_{>0}^{S(H)}) \subseteq \mathbb{R}_{>0}^{S(H)}.$$

*Proof.* Since  $S(H) \subseteq S(u)$  for each  $u \in H$ , we have  $\mathbb{R}_{>0}^{S(H)} \subseteq \mathbb{R}_{>0}^{S(u)}$ . By forward invariance,  $T_e(\mathbb{R}_{>0}^{S(u)}) \subseteq \mathbb{R}_{>0}^{S(v)}$ . Because  $S(H) \subseteq S(v)$ , restricting to coordinates in  $S(H)$  yields invariance on  $S(H)$ .  $\square$

### 13.3 Support-connectivity on the SCC face as a finite graph property

Fix an SCC  $H$  and its canonical face  $S(H)$ . For each generator edge  $e$  in  $H$  labeled by  $T_e$ , define the coordinate support digraph on  $S(H)$ : we draw an arrow  $j \rightarrow i$  if there exists an edge  $e$  in  $H$  with  $(T_e)_{ij} > 0$ . Thus  $j \rightarrow i$  records that coordinate  $j$  can contribute to coordinate  $i$  in one admissible step inside  $H$ .

**Definition 13.4** (Coordinate support digraph on a face). *Let  $\Gamma(H)$  be the directed graph with vertex set  $S(H)$  and an arrow  $j \rightarrow i$  if there exists an edge  $e$  in  $H$  such that  $(T_e)_{ij} > 0$ .*

**Lemma 13.5** (Support-connectivity certificate). *If  $\Gamma(H)$  is strongly connected, then for every  $i, j \in S(H)$  there exists an admissible word  $w$  in  $H$  such that the block product satisfies  $(B(w))_{ij} > 0$ .*

*Proof.* Strong connectivity gives a coordinate path  $j \rightarrow \dots \rightarrow i$  realized by a sequence of generator matrices from edges of  $H$ . Concatenate the corresponding edges to obtain a word  $w$  whose product contains a positive contribution from  $j$  into  $i$ .  $\square$

**Corollary 13.6** (Doeblin word from aperiodic SCC via Boolean-pattern product automaton). *Let  $H$  be an aperiodic SCC of the tight graph  $G^{(0)}$  and assume a consistent face map  $S(\cdot)$  is available, with active face  $S(H) \neq \emptyset$ . Then the following are equivalent:*

- (i) *There exists a finite word  $w$  supported in  $H$  such that the block product  $B(w)$  is strictly positive on  $S(H) \times S(H)$  (equivalently,  $w$  is Doeblin on the face  $S(H)$ ).*
- (ii) *The Boolean-pattern product-automaton test (Algorithm 7) reaches a return state  $(v_0, \mathbf{1})$  for some base vertex  $v_0 \in H$ , where  $\mathbf{1}$  denotes the all-ones pattern on  $S(H) \times S(H)$ ; in this case the algorithm returns an explicit witness word  $w$ .*

*In case (i), strict positivity of  $B(w)_{S(H) \times S(H)}$  implies that for every  $i, j \in S(H)$  there exists a directed coordinate path  $j \rightarrow \dots \rightarrow i$  in  $\Gamma(H)$ , hence  $\Gamma(H)$  is strongly connected. Thus failure of strong connectivity produces the OG3 cut witness of Lemma 13.10, and strong connectivity of  $\Gamma(H)$  is a fast necessary check. When  $\Gamma(H)$  is strongly connected, the existence of a Doeblin word is decided by the Boolean-pattern product-automaton test (Algorithm 7); the graph condition alone does not certify strict positivity on  $S(H) \times S(H)$ .*

**Remark 13.7** (Strong connectivity is necessary but not sufficient). *Strong connectivity of  $\Gamma(H)$  is a fast necessary test for the existence of a Doeblin word. However,  $\Gamma(H)$  records only one-step support reachability and does not encode higher-order pattern constraints that can prevent simultaneous positivity of all pairs. A clean counter-shape is the permutation case: if all allowed face-restricted patterns are permutation matrices generating a transitive action on  $S(H)$ , then  $\Gamma(H)$  is strongly connected while every admissible product remains a permutation pattern, so no word yields strict positivity on  $S(H) \times S(H)$ . For this reason, the pipeline uses the Boolean-pattern product-automaton test (Algorithm 7) as the definitive finite decision procedure; when it fails, it returns an explicit finite non-reachability certificate in that automaton.*

### 13.4 Algorithmic extraction

### 13.5 Witness extraction when certificates fail

The obstruction predicates OG2/OG3 are designed to be not merely *named* failures but *witness-producing* ones. Here we record explicit finite witnesses that can be returned whenever the face-map or coordinate-connectivity checks fail.

**Definition 13.8** (Support propagation operator). *For an edge  $e : u \rightarrow v$  labeled by a nonnegative template/block  $T_e$  and a set  $S \subseteq \{1, \dots, d\}$ , define the propagated support*

$$\text{Prop}_e(S) := \{i : \exists j \in S \text{ with } (T_e)_{ij} > 0\}.$$

*For a word  $w = e_1 \dots e_L$  we set  $\text{Prop}_w := \text{Prop}_{e_L} \circ \dots \circ \text{Prop}_{e_1}$ .*

**Lemma 13.9** (OG2 witness: finite inconsistency cycle). *Let  $H$  be an SCC and suppose the proposed face sets  $S(v)$  fail to form a face map on  $H$ . Then there exist vertices  $u, v \in H$ , an edge  $e : u \rightarrow v$  in  $H$ , and indices  $j \in S(u)$  and  $i \notin S(v)$  such that  $(T_e)_{ij} > 0$ . Moreover, choosing any directed path  $p$  in  $H$  from  $v$  back to  $u$  and setting  $w := ep$ , we obtain a directed cycle word  $w$  in  $H$  together with a marked step (the first step  $e$ ) at which a coordinate outside the designated face is activated. This pair  $(w, (i, j))$  is a finite witness of OG2 on  $H$ .*

*Proof.* If  $S(\cdot)$  is not a face map on  $H$ , then some edge  $e : u \rightarrow v$  in  $H$  violates Definition 13.1. In particular, the “no spurious activation” condition (ii) fails, which means exactly that  $\text{Prop}_e(S(u)) \not\subseteq S(v)$ . Equivalently, there exist  $j \in S(u)$  and  $i \notin S(v)$  with  $(T_e)_{ij} > 0$ . Because  $H$  is strongly connected, there exists a directed path  $p$  from  $v$  to  $u$ , so  $w := ep$  is a directed cycle word in  $H$ . The marked first step  $e$  certifies the activation of an outside coordinate along a cycle in  $H$ .  $\square$

A portable machine witness for Lemma 13.9 is included in this repository:

- `og2_inconsistency_witness_schema.json`
- `og2_inconsistency_witness_example.json`

**Lemma 13.10** (OG3 witness: cut decomposition of  $\Gamma(H)$ ). *Let  $H$  be a tight SCC for which a face map exists and let  $S(H)$  be the SCC face. If the coordinate digraph  $\Gamma(H)$  is not strongly connected, then there exist nonempty disjoint sets  $A, B \subseteq S(H)$  with  $A \cup B = S(H)$  such that*

$$(T_e)_{ij} = 0 \quad \text{for every edge } e \text{ in } H, \text{ every } i \in A, \text{ and every } j \in B.$$

*Equivalently,  $\Gamma(H)$  has no directed edges from  $B$  to  $A$ . In particular, for every admissible word  $w$  in  $H$  and every  $i \in A, j \in B$  we have  $(B(w))_{ij} = 0$ . Thus  $(H, S(H), A, B)$  is a finite witness of OG3.*

*Proof.* Compute the SCC decomposition of  $\Gamma(H)$  and consider its condensation DAG. Since  $\Gamma(H)$  is not strongly connected, this DAG has at least two nodes and hence has a sink SCC  $A$ . Let  $B := S(H) \setminus A$ . By the definition of a sink SCC, there is no directed edge in  $\Gamma(H)$  from any vertex in  $B$  to any vertex in  $A$ . By Definition 13.4, an edge  $j \rightarrow i$  in  $\Gamma(H)$  exists exactly when some generator edge  $e$  in  $H$  has  $(T_e)_{ij} > 0$ . Therefore for all  $i \in A$  and  $j \in B$ , and for all edges  $e$  in  $H$ , we have  $(T_e)_{ij} = 0$ . The final structural-zero claim for products follows by closure under multiplication of nonnegative matrices: if every generator has zeros on  $A \times B$ , then every product does as well.  $\square$

A portable machine witness for Lemma 13.10 is included in this repository:

- `og3_cut_witness_schema.json`
- `og3_cut_witness_example.json`

When  $\Gamma(H)$  is strongly connected but no Doeblin word exists, the product-automaton procedure returns a finite nonreachability certificate; a schema and minimal example are included:

- `og3_nonreachability_certificate_schema.json`
- `og3_nonreachability_certificate_example.json`

---

**Algorithm 5** Compute SCC face and coordinate support graph

---

**Require:** SCC  $H$  with face map  $S(\cdot)$  and generator labels  $T_e$

**Ensure:**  $S(H)$  and coordinate digraph  $\Gamma(H)$

- 1:  $S(H) \leftarrow \bigcap_{v \in H} S(v)$
  - 2: Initialize  $\Gamma(H)$  with vertex set  $S(H)$
  - 3: **for all** edges  $e$  inside  $H$  **do**
  - 4:     **for all**  $i, j \in S(H)$  **do**
  - 5:         **if**  $(T_e)_{ij} > 0$  **then** add arrow  $j \rightarrow i$
  - 6:         **end if**
  - 7:     **end for**
  - 8: **end for**
-

---

**Algorithm 6** Check OG2–OG3 and produce Doeblin/recurrence witnesses

---

**Require:** Tight graph  $G^{(0)}$ , SCC decomposition, period and coordinate digraphs  $\Gamma(H)$

**Ensure:** Either a finite obstruction witness (OG1/OG2/OG3), or a Doeblin witness  $w$  together with either a syndetic modulus  $L_{\text{syn}}$  (RC1) or an explicit avoidance-cycle witness

```
1: Compute SCCs of  $G^{(0)}$  and their periods  $\text{per}(H)$ 
2: for each aperiodic SCC  $H$  do
3:   Attempt to compute a consistent face map  $S(\cdot)$  on  $H$ 
4:   if inconsistent then
5:     return OG2 witness (inconsistency cycle)
6:   end if
7:   Build  $\Gamma(H)$  on  $S(H)$ 
8:   if  $\Gamma(H)$  is not strongly connected then
9:     return OG3 witness of type (cut) via Lemma 13.10
10:  end if
11:  Run the product-automaton Doeblin-word test (Algorithm 7)
12:  if Algorithm 7 returns a word  $w$  then
13:    Build the avoidance graph  $\mathcal{A}_{\neg w}(H)$  (Definition 15.2)
14:    if  $\mathcal{A}_{\neg w}(H)$  contains a directed cycle then
15:      return Doeblin witness  $w$  and an avoidance-cycle witness (Lemma 15.3)
16:    else
17:      Compute  $L_{\text{syn}} = 1 + \ell_{\max}$  where  $\ell_{\max}$  is the longest path length in  $\mathcal{A}_{\neg w}(H)$ 
18:      return  $(w, L_{\text{syn}})$  (RC1; Lemma 15.6)
19:    end if
20:  else
21:    return OG3 witness of type (product-automaton): a finite non-reachability certificate
    in the Boolean-pattern product automaton (no return state  $(v_0, \mathbf{1})$  is reachable)
22:  end if
23: end for
24: return OG1 witness (no aperiodic SCC exists in  $G^{(0)}$ )
```

---

## 14 Doeblin-word discovery by a finite product automaton

This section records the finite-state procedure used throughout the certificate pipeline to decide whether a given aperiodic tight SCC admits a Doeblin word on its active face. The input is purely combinatorial support information on the face, and the output is either a concrete witness word or a finite non-reachability certificate.

### 14.1 Boolean patterns and Boolean product

Fix a tight SCC  $H$  and a face  $S = S(H) \subseteq \{1, \dots, d\}$ . For each tight edge  $e : u \rightarrow v$  in  $H$ , let  $B_e \in \mathbb{R}_+^{d \times d}$  denote the corresponding nonnegative block (or fixed-length template). Define the face-restricted support pattern

$$P_e \in \{0, 1\}^{S \times S}, \quad (P_e)_{ij} = 1 \iff (B_e)_{ij} > 0 \text{ for } i, j \in S.$$

For Boolean patterns  $P, Q \in \{0, 1\}^{S \times S}$ , define their Boolean product  $P \odot Q$  by

$$(P \odot Q)_{ij} = \bigvee_{k \in S} (P_{ik} \wedge Q_{kj}),$$

so that  $P \odot Q$  records the support pattern of the matrix product of the underlying nonnegative blocks restricted to  $S$ .

## 14.2 The product automaton and BFS search

Let  $\mathcal{P} := \{0, 1\}^{S \times S}$  be the finite set of Boolean patterns. The product automaton has state space  $H \times \mathcal{P}$ . A directed transition

$$(u, P) \xrightarrow{e: u \rightarrow v} (v, P_e \odot P)$$

records the effect of appending the edge  $e$  to the front of the word whose current pattern is  $P$ .

---

**Algorithm 7** Doeblin-word discovery by finite product automaton

---

**Require:** Tight SCC  $H$ , active face  $S(H)$ , Boolean support patterns  $P_e$  on  $S(H)$  for each tight edge  $e$  in  $H$

**Ensure:** Either a word  $w$  whose face-restricted product is strictly positive, or a finite non-reachability certificate

```

1: Choose a base vertex  $v_0 \in H$ 
2: Let  $I$  be the identity pattern on  $S(H)$  and let  $\mathbf{1}$  denote the all-ones pattern on  $S(H) \times S(H)$ 
3: Initialize BFS on states  $(v, P)$  with the start state  $(v_0, I)$ 
4: while queue not empty do
5:   Pop  $(u, P)$ 
6:   for all tight edges  $e : u \rightarrow v$  in  $H$  do
7:      $P' \leftarrow P_e \odot P$ 
8:     if  $(v, P')$  not yet visited then
9:       Mark visited; record predecessor; enqueue  $(v, P')$ 
10:    end if
11:    if  $v = v_0$  and  $P' = \mathbf{1}$  then
12:      Reconstruct the witnessed word  $w$  from predecessors and return  $w$ 
13:    end if
14:  end for
15: end while
16: return finite non-reachability certificate:  $\mathbf{1}$  is not reachable at  $(v_0, \cdot)$ 

```

---

**Definition 14.1** (Forward-closed nonreachability certificate). *A set  $\mathcal{F} \subseteq H \times \mathcal{P}$  is called forward-closed if whenever  $(u, P) \in \mathcal{F}$  and*

$$(u, P) \xrightarrow{e: u \rightarrow v} (v, P_e \odot P)$$

*is a transition in the product automaton, then  $(v, P_e \odot P) \in \mathcal{F}$ . If  $\mathcal{F}$  is forward-closed, contains the start state  $(v_0, I)$ , and does not contain the target state  $(v_0, \mathbf{1})$ , then  $\mathcal{F}$  is a valid nonreachability certificate for the Doeblin goal. This is the content serialized by the field `forward_closed_set` in the OG3 nonreachability artifact schema.*

**Lemma 14.2** (Correctness and completeness of Algorithm 7). *Fix a tight SCC  $H$  and active face  $S(H)$  with Boolean patterns  $P_e$ . Algorithm 7 terminates and satisfies the following dichotomy:*

- *If it returns a word  $w$ , then the face-restricted pattern of the tight product along  $w$  is  $\mathbf{1}$ , hence  $w$  is a Doeblin word on  $S(H)$ .*

- If it returns a nonreachability certificate, then no Doeblin word on  $S(H)$  exists within the tight language of  $H$ . Equivalently, the goal state  $(v_0, \mathbf{1})$  is unreachable from  $(v_0, I)$  in the product automaton.

*Proof.* The product automaton has finite state space  $H \times \mathcal{P}$ , so the BFS terminates.

If the algorithm returns a word  $w$ , it has found a closed walk from  $(v_0, I)$  to  $(v_0, \mathbf{1})$  labeled by a tight-edge word. By construction of transitions, the accumulated Boolean product along that word equals  $\mathbf{1}$ , which is exactly the definition of a Doeblin word on the face.

If the algorithm exhausts the BFS queue without reaching  $(v_0, \mathbf{1})$ , let  $\mathcal{F}$  be the set of visited states. By BFS construction,  $\mathcal{F}$  contains  $(v_0, I)$  and is forward-closed under transitions, hence it is a certificate in the sense of Definition 14.1. Therefore  $(v_0, \mathbf{1})$  is unreachable from  $(v_0, I)$ . Any Doeblin word would produce a labeled path from  $(v_0, I)$  to  $(v_0, \mathbf{1})$ , contradiction.  $\square$

**Remark 14.3** (Complexity). *The state space has size  $|H| \cdot 2^{|S(H)|^2}$ . This exponential dependence on the face size is unavoidable at the level of brute-force semigroup reachability, but in the intended regime the active faces are descriptor-controlled and small.*

**Remark 14.4** (Language-aware reachability versus conservative support filters). *In some implementations it is convenient to precompute auxiliary reachability graphs from template support information that are (intentionally) conservative. Such over-approximations may make the presentation look looser than necessary, but they do not affect correctness: the decisive Doeblin step is the product-automaton decision above, which is built from the actual tight-language edges in  $H$  and the face-restricted patterns  $P_e$ . Any word returned by Algorithm 7 is a valid witness on the declared face, and any non-reachability conclusion is only asserted for the declared automaton.*

## 15 Syndetic return to a certified Doeblin word via an avoidance graph

Fix a tight SCC  $H$  and a word  $w = e_1 \cdots e_m$  on the edges of  $H$ . This section formalizes a finite, checkable recurrence condition for  $w$ . The outcome is dichotomous:

- either the avoidance dynamics contains a directed cycle, yielding an explicit witness that  $w$  can be avoided for arbitrarily long times,
- or the avoidance dynamics is acyclic, in which case  $w$  must occur with bounded gaps along every infinite trajectory in  $H$ .

The second case is the concrete recurrence input (RC1) used in the gap-free contraction statements.

**Definition 15.1** (Finite recurrence input (RC1)). *Let  $H$  be a tight SCC and let  $w$  be an edge word in  $H$ . We say that  $(H, w)$  satisfies **(RC1)** if the avoidance graph  $A(H, w)$  (Definition 15.2) is acyclic. In this case Algorithm 8 returns a finite modulus  $L_{\text{syn}}(H, w)$  such that every edge word of length  $L_{\text{syn}}(H, w)$  in  $H$  contains an occurrence of  $w$ . For notational compatibility with later sections, we also write  $\mathcal{A}_{\neg w}(H) := A(H, w)$ .*

### 15.1 Prefix automaton for a fixed word

Let  $m := |w|$ . For a finite edge word  $a = e'_1 \cdots e'_k$  in  $H$ , define  $\pi(a) \in \{0, 1, \dots, m-1\}$  to be the length of the longest suffix of  $a$  which is a proper prefix of  $w$ . Equivalently,  $\pi(a) = q$  means that

the last  $q$  edges of a match  $e_1 \cdots e_q$ , but  $a$  does not yet contain a full occurrence of  $w$  ending at the last edge.

Given  $q \in \{0, \dots, m-1\}$  and an edge  $e$  of  $H$ , define the update  $\text{next}(q, e) \in \{0, \dots, m\}$  to be the new matched-prefix length after appending  $e$ . If  $\text{next}(q, e) = m$ , then  $w$  occurs ending at that step. Otherwise  $\text{next}(q, e) \in \{0, \dots, m-1\}$ . This is the standard prefix automaton update (KMP-type) for a fixed pattern.

## 15.2 Avoidance graph

**Definition 15.2** (Avoidance graph). *Let  $A(H, w)$  be the directed graph on vertex set*

$$\{0, 1, \dots, m-1\} \times V(H).$$

*There is a directed edge*

$$(q, x) \rightarrow (q', y)$$

*in  $A(H, w)$  if and only if there exists an edge  $e : x \rightarrow y$  of  $H$  such that  $\text{next}(q, e) = q' \in \{0, \dots, m-1\}$ . In other words,  $A(H, w)$  encodes the evolution of the matched-prefix state along paths in  $H$  under the constraint that no step completes an occurrence of  $w$ .*

**Lemma 15.3** (Avoidance-cycle witness). *If  $A(H, w)$  contains a directed cycle, then there exists an infinite trajectory in  $H$  which avoids  $w$  forever. Consequently, no bound on gaps between occurrences of  $w$  can hold uniformly over all infinite trajectories in  $H$ .*

*Proof.* A directed cycle in  $A(H, w)$  corresponds to a nonempty edge word in  $H$  which returns to the same matched-prefix state without ever completing  $w$ . Repeating this cycle yields an infinite path in  $H$  for which the prefix-state never reaches  $m$ , hence  $w$  never occurs.  $\square$

## 15.3 Sofic refinement: the $w$ -avoidance shift

The avoidance graph also gives a finite-state symbolic model for the “bad” itineraries that avoid the certified word.

**Definition 15.4** ( $w$ -avoidance shift). *Let  $X_{\neg w}(H) \subset E(H)^{\mathbb{Z}}$  be the set of bi-infinite edge sequences in  $H$  that avoid  $w$  at every time. Equivalently,  $X_{\neg w}(H)$  is the two-sided edge shift of the avoidance graph  $A(H, w)$ , viewed as a labeled graph by the underlying edges of  $H$ . In particular,  $X_{\neg w}(H)$  is a sofic shift presented by a finite graph.*

**Proposition 15.5** (RC1 versus nonemptiness of the avoidance shift). *For fixed  $(H, w)$ , the following are equivalent:*

- $A(H, w)$  contains a directed cycle;
- $X_{\neg w}(H)$  is nonempty;
- $OG4(H, w)$  holds (equivalently, RC1 fails).

*Consequently, RC1 holds if and only if  $X_{\neg w}(H) = \emptyset$ , i.e. there are no bi-infinite tight itineraries in  $H$  that avoid  $w$ .*

*Proof.* A finite directed graph supports a bi-infinite path if and only if it contains a directed cycle. By Definition 15.4, bi-infinite  $w$ -avoiding itineraries in  $H$  correspond exactly to bi-infinite paths in  $A(H, w)$ . The remaining equivalences are Definitions 15.1 and 18.1.  $\square$

**Lemma 15.6** (Syndetic modulus from acyclicity). *If  $A(H, w)$  is acyclic, then there exists a finite integer  $L_{\text{syn}}(H, w)$  such that every length- $L_{\text{syn}}(H, w)$  edge word in  $H$  contains an occurrence of  $w$ . In particular, every infinite trajectory in  $H$  contains occurrences of  $w$  with gaps at most  $L_{\text{syn}}(H, w)$ .*

*Proof.* If  $A(H, w)$  is acyclic, it has finite longest directed path length. Along any edge word in  $H$  that avoids  $w$ , the corresponding lifted path in  $A(H, w)$  never hits an accepting step ( $\text{next} = m$ ), so it remains a directed path in the avoidance graph. Therefore any  $w$ -avoiding word has length at most the longest path length of  $A(H, w)$ . Taking  $L_{\text{syn}}(H, w)$  to be one plus that maximal length forces every longer word to contain  $w$ .  $\square$

**Corollary 15.7** (Explicit recurrence bound from automaton size). *If  $A(H, w)$  is acyclic and has  $N$  vertices, then one may take*

$$L_{\text{syn}}(H, w) \leq N.$$

*In particular, since  $N = m |V(H)|$ , the bounded-gap modulus can be chosen at most  $m |V(H)|$ .*

*Proof.* An acyclic directed graph on  $N$  vertices has no directed path longer than  $N - 1$  edges, so  $\ell_{\text{max}} \leq N - 1$  in Algorithm 8. Hence  $L_{\text{syn}} = \ell_{\text{max}} + 1 \leq N$ .  $\square$

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**Algorithm 8** Syndetic return check and modulus computation

---

**Require:** Tight SCC  $H$ , word  $w$  on edges of  $H$  of length  $m$

**Ensure:** Either an avoidance-cycle witness, or a syndetic modulus  $L_{\text{syn}}(H, w)$

- 1: Build the avoidance graph  $A(H, w)$  (Definition 15.2)
  - 2: **if**  $A(H, w)$  contains a directed cycle **then**
  - 3:     Extract a directed cycle as a witness and **return** avoidance-cycle witness
  - 4: **else**
  - 5:     Compute the length  $\ell_{\text{max}}$  of a longest directed path in  $A(H, w)$  (DP on a topological ordering)
  - 6:     **return**  $L_{\text{syn}}(H, w) \leftarrow \ell_{\text{max}} + 1$
  - 7: **end if**
- 

## 16 Effective extremal language and a sofic Mather model

This section upgrades the “effective extremal itinerary” discussion to a fully witness-producing statement. The output is a finite symbolic object that can be validated independently, and it is the artifact associated with OP-MATHER-EFF.

### 16.1 From calibrated minimizers to a finite boundary carrier

Recall that canonical normalization and calibration assign reduced costs  $c_\phi(e) \geq 0$  to edges and define the tight subgraph  $G^{(0)}$  consisting of edges with  $c_\phi(e) = 0$ . The boundary carrier  $\Sigma_{\mathcal{A}}$  (Definition 6.6) is the two-sided edge shift of a finite directed graph  $G^{\mathcal{A}}$  extracted from the reduction.

**Proposition 16.1** (Computable boundary carrier). *Given a certified instance (Definition 22.1), the boundary carrier  $\Sigma_{\mathcal{A}}$  is computable as a finite shift of finite type:*

- the underlying graph  $G^{\mathcal{A}}$  is obtained by selecting the tight edges and restricting to the active face data,
- the irreducible components of  $\Sigma_{\mathcal{A}}$  are the SCC edge shifts of  $G^{\mathcal{A}}$ .

Moreover, in the periodic regime of Proposition 9.3, each component is a single periodic orbit.

*Proof.* All objects in the construction are finite. The tight edge test is a finite comparison of reduced costs. Face restriction is a finite support check on the template blocks restricted to the declared active support sets. SCC decomposition is a finite graph computation. The final statement follows from Proposition 9.3.  $\square$

## 16.2 Sofic refinement from a Doeblin word and recurrence/avoidance

Fix a tight SCC  $H \subseteq G^A$ . If  $w$  is a word supported in  $H$ , then the  $w$ -avoidance shift  $X_{\neg w}(H)$  (Definition 15.4) is accepted by the finite avoidance automaton  $A(H, w)$ . In particular,  $X_{\neg w}(H) \neq \emptyset$  is witnessed by a directed cycle in  $A(H, w)$ .

**Definition 16.2** (Sofic extremal model associated to  $(H, w)$ ). *Let  $H$  be a tight SCC and let  $w$  be a word supported in  $H$ . The sofic extremal model is the finite symbolic package consisting of:*

- the edge shift  $X_H$  of  $H$  (a shift of finite type given by the adjacency of  $H$ ),
- the avoidance automaton  $A(H, w)$ ,
- either an avoidance-cycle witness (a directed cycle in  $A(H, w)$ ), or a bounded-gap modulus  $B(H, w)$  proving that every bi-infinite path in  $H$  contains an occurrence of  $w$  in every window of length  $B(H, w)$ .

The bounded-gap modulus is the recurrence certificate RC1: when  $A(H, w)$  is acyclic, a finite longest-path computation yields  $B(H, w)$ .

**Theorem 16.3** (Effective Mather/sofic description: periodic vs aperiodic regimes). *For any certified instance, the pipeline returns a finite extremal itinerary model as follows.*

- **Periodic regime.** *If the minimizing carrier is minimal, then every tight SCC is a directed cycle (Proposition 9.3). In this case the extremal itinerary set is a finite union of periodic orbits, each specified by its cycle in  $G^A$ .*
- **Aperiodic regime with a candidate core.** *If  $H$  is an aperiodic tight SCC and  $w$  is a certified Doeblin word on the associated face block, then the recurrence layer decides which of the following holds:*
  - **RC1 holds:**  $A(H, w)$  is acyclic, a bounded-gap modulus  $B(H, w)$  is computed, and every tight itinerary in  $H$  contains  $w$  syndetically.
  - **OG4 holds:** a directed cycle in  $A(H, w)$  is returned, yielding a periodic tight itinerary that avoids  $w$  forever.

*In either subcase, the output is a concrete sofic extremal model in the sense of Definition 16.2, and it is verifiable directly from the returned finite data.*

*Consequently, OP-MATHER-EFF holds on the certified class: extremal itineraries admit an explicit finite-state model computed by the pipeline, and failure of the desired recurrence is witnessed by a finite avoidance cycle.*

*Proof.* The periodic regime statement is Proposition 9.3. For the aperiodic regime, the Doeblin-word search (Section 14) produces either a word  $w$  with a positivity margin on the face block or a finite failure witness (OG3). Given  $(H, w)$ , the avoidance automaton  $A(H, w)$  is finite and its cycle/acyclic dichotomy is decided by finite graph search. If acyclic, the maximal path length is finite and gives  $B(H, w)$ ; if cyclic, the cycle itself yields a periodic  $w$ -avoiding itinerary. All returned objects are finite and checkable.  $\square$

### 16.3 Artifact: a self-contained symbolic model object

The paper includes a lightweight JSON schema for exporting the model in Definition 16.2. The export schema is recorded in `mather_sofic_model_schema.json` and documented in Section 17. A verifier for the object is straightforward: validate the schema, then check that each listed transition belongs to the declared tight SCC, and that the avoidance-cycle witness (if present) is a genuine directed cycle.

## 17 Schema: MatherSoficModel (OP–MATHER–EFF artifact)

This section records a lightweight export schema for the sofic extremal model of Definition 16.2. The object is designed to be self-contained: it describes a tight SCC shift  $X_H$ , the avoidance automaton for a chosen word  $w$ , and the finite witness concluding either RC1 (bounded-gap recurrence) or OG4 (an avoidance cycle).

**Files.** The JSON schema is `mather_sofic_model_schema.json`.

**Core fields.**

- **core:** basic finite data for the tight SCC shift  $X_H$  (counts and an explicit transition list), with optional **face\_support**.
- **avoidance\_automaton:** state and edge counts for the avoidance automaton  $A(H, w)$ .
- **outcome:** either RC1 (with **gap\_bound\_B**) or OG4 (with an explicit **avoidance\_cycle**).

**Verification.** A verifier checks:

- schema validity,
- that each listed transition is an edge of the declared SCC shift,
- if RC1: that the reported **gap\_bound\_B** is consistent with an acyclicity claim for the avoidance graph (when the full graph is provided by reference),
- if OG4: that **avoidance\_cycle** forms a directed cycle in the avoidance graph.

In the full pipeline, the needed supporting graphs are already present in the certificate or obstruction witness payloads; the model object acts as a compact symbolic summary.

## 18 Recurrence Obstruction: Doeblin-Avoidance Cycle (RC1 Failure)

The obstruction taxonomy OG1–OG3 controls the existence of an aperiodic tight SCC, a consistent active-face map, and a strictly positive Doeblin block on that active face. Even when none of these obstructions occurs, a logically correct gap-free contraction statement must still address one further issue: a certified Doeblin word need not recur along *every* tight trajectory unless its recurrence is forced by finite-state structure.

We therefore isolate a finite recurrence predicate (**RC1**) for a fixed SCC and a fixed Doeblin word, and we record its failure as a fourth obstruction class OG4 with an explicit finite witness. This keeps recurrence out of the hypotheses while preserving a complete, decidable boundary taxonomy.

## 18.1 Definition and finite witness schema

**Definition 18.1** (OG4 / recurrence obstruction: Doeblin-avoidance cycle). *Let  $H$  be a tight SCC of  $G^{(0)}$  and let  $w$  be a finite edge-word supported in  $H$ . We say that the obstruction  $\mathbf{OG4}(H, w)$  holds if the avoidance graph  $\mathcal{A}_{\neg w}(H)$  (Definition 15.2) contains a directed cycle. Equivalently,  $(\mathbf{RC1})$  fails for  $(H, w)$  (Definition 15.1).*

A finite witness for  $\mathbf{OG4}(H, w)$  is simply a directed cycle in  $\mathcal{A}_{\neg w}(H)$ , which can be produced by standard cycle-detection on a finite directed graph.

The pipeline records obstruction witnesses as a portable JSON object. A concrete schema and a minimal example are included alongside this paper:

- `obstruction_witness_schema.json`
- `obstruction_witness_example.json`

## 18.2 Why OG4 is a genuine obstruction

**Proposition 18.2** (OG4 produces a tight minimizing trajectory with no Doeblin blocks). *If  $\mathbf{OG4}(H, w)$  holds, then there exists a periodic tight path  $\pi$  supported in  $H$  that avoids  $w$  entirely. In particular, no theorem can guarantee occurrence of  $w$  along all tight minimizers in  $H$ .*

*Proof.* By Definition 18.1,  $\mathcal{A}_{\neg w}(H)$  contains a directed cycle. Lemma 15.3 converts such a cycle into an explicit periodic tight path in  $H$  that avoids  $w$  entirely. Since the path is tight, Lemma 6.7 shows that it has asymptotic mean cost  $\lambda_*$ , so it is a minimizing trajectory, yet it contains no occurrences of  $w$ .  $\square$

**Remark 18.3** (Relationship to OG1–OG3). *OG4 can occur even when OG1–OG3 do not: an aperiodic tight SCC may exist, the face map may be consistent, and a strictly positive Doeblin word  $w$  may be certified, but  $w$  may still be avoidable by some tight trajectories in  $H$ . Accordingly, OG4 is the correct finite obstruction to record whenever  $(\mathbf{RC1})$  is not satisfied for the chosen  $(H, w)$ .*

## 19 Contraction from a Certified Doeblin Word: the Finite Recurrence Branch

Once the finite obstruction taxonomy produces a certified Doeblin word on an aperiodic tight SCC, there is only one further way for “Doeblin  $\Rightarrow$  contraction” to fail on *all* tight trajectories: the Doeblin word might be avoidable. This is exactly OG4, witnessed by a directed cycle in the avoidance graph. When OG4 does not occur, the Doeblin word is *unavoidable* on the induced subshift over the SCC, hence appears with bounded gaps on every tight path, and uniform projective contraction follows with fully explicit constants.

We also record a second, quantitative route: along a general minimizing path, bounded gaps need not hold globally because tightness holds only in density. Nevertheless, if the minimizer spends positive lower density of tight time in the aperiodic SCC, then unavoidable recurrence forces a uniform positive lower density of Doeblin blocks, which is enough for an explicit exponential contraction rate.

## 19.1 Hilbert metric and Birkhoff contraction

**Definition 19.1** (Hilbert (projective) metric). For  $x, y \in \mathbb{R}_{>0}^S$ , define

$$d_H(x, y) := \log\left(\max_{i \in S} \frac{x_i}{y_i}\right) + \log\left(\max_{i \in S} \frac{y_i}{x_i}\right) = \log\left(\max_{i, j \in S} \frac{x_i y_j}{x_j y_i}\right).$$

This metric is invariant under positive scalar rescaling:  $d_H(cx, dy) = d_H(x, y)$  for all  $c, d > 0$ .

**Lemma 19.2** (Birkhoff contraction from entry bounds). Let  $A \in \mathbb{R}_{>0}^{S \times S}$  satisfy  $\min_{i, j \in S} A_{ij} \geq \varepsilon$  and  $\max_{i, j \in S} A_{ij} \leq U$ . Set

$$\Delta := 2 \log \frac{U}{\varepsilon}, \quad \tau := \tanh\left(\frac{\Delta}{4}\right) = \tanh\left(\frac{1}{2} \log \frac{U}{\varepsilon}\right) \in (0, 1).$$

Then for all  $x, y \in \mathbb{R}_{>0}^S$ ,

$$d_H(Ax, Ay) \leq \tau d_H(x, y).$$

**Remark 19.3.** The constant  $\tau$  is the standard Birkhoff contraction coefficient corresponding to the projective diameter bound  $\Delta$  induced by entry ratio control; see Birkhoff [11] and modern treatments such as Bushell [12] or Lemmens–Nussbaum [13]. The certificate pipeline will provide explicit values of  $(\varepsilon, U)$  for Doeblin words.

**Definition 19.4** (Recurrence input for a Doeblin word). Fix an infinite tight path  $\pi$  in a tight SCC  $H$  and a Doeblin word  $w$  of length  $|w|$  on  $H$ . Let  $N_w(n; \pi)$  denote the number of occurrences of  $w$  starting among the first  $n$  edges of  $\pi$ . Let  $N_w^{\text{dis}}(n; \pi)$  denote the maximal number of pairwise disjoint occurrences of  $w$  whose starting indices lie in  $\{0, \dots, n-1\}$ ; equivalently, the greedy selection rule yields  $N_w^{\text{dis}}(n; \pi) \geq \lfloor N_w(n; \pi)/|w| \rfloor$ . We say that  $\pi$  satisfies one of the following recurrence inputs.

- **(RH1) Syndetic recurrence:** there exists  $L_0 \in \mathbb{N}$  such that every length- $L_0$  subword of  $\pi$  contains an occurrence of  $w$ .
- **(RH2) Positive lower density:** there exists  $\rho > 0$  such that  $\liminf_{n \rightarrow \infty} \frac{N_w(n; \pi)}{n} \geq \rho$ .

**Corollary 19.5** (Exponential projective contraction from Doeblin blocks). Let  $w$  be a Doeblin word on a face  $S$  whose face-restricted product  $B(w)|_S$  has entry bounds  $(\varepsilon, U)$ . Let  $\tau \in (0, 1)$  be the corresponding Birkhoff coefficient from Lemma 19.2. Then along any tight path  $\pi$  in  $H$ , the Hilbert distance between two positive face vectors transported by the tight product contracts exponentially provided  $\pi$  satisfies either recurrence input from Definition 19.4. More precisely:

- Under **(RH1)** with modulus  $L_0$ , one has  $d_H(x_n, y_n) \leq \tau^{\lfloor n/L_0 \rfloor} d_H(x_0, y_0)$  and hence  $d_H(x_n, y_n) \leq C e^{-\kappa n} d_H(x_0, y_0)$  with  $\kappa := \frac{1}{L_0}(-\log \tau)$  and  $C := \tau^{-1}$ .
- Under **(RH2)** with density  $\rho$ , let  $N_w^{\text{dis}}(n; \pi)$  be as in Definition 19.4. Then  $d_H(x_n, y_n) \leq \tau^{N_w^{\text{dis}}(n; \pi)} d_H(x_0, y_0)$ . Moreover, extracting disjoint occurrences by the greedy rule gives  $N_w^{\text{dis}}(n; \pi) \geq \lfloor N_w(n; \pi)/|w| \rfloor$ , so the density bound implies an explicit exponential form  $d_H(x_n, y_n) \leq C e^{-\kappa n} d_H(x_0, y_0)$  with  $\kappa := \frac{\rho}{|w|}(-\log \tau)$  and  $C := \tau^{-1}$ .

## 19.2 A tight-path version (uniform on the induced subshift over the tight SCC)

**Theorem 19.6** (Finite recurrence dichotomy and contraction on tight paths). Fix an aperiodic tight SCC  $H$  of  $G^{(0)}$  and assume OG2 and OG3 do not occur on  $H$ , so that a consistent active face  $S(H)$  exists and an explicit Doeblin word  $w$  supported in  $H$  is certified with  $B(w)$  strictly positive on  $S(H) \times S(H)$ .

Then exactly one of the following holds.

(OG4) The avoidance graph  $\mathcal{A}_{\neg w}(H)$  contains a directed cycle. Equivalently, there exists a periodic tight path in  $H$  that avoids  $w$  entirely (Proposition 18.2).

(UC) The avoidance graph  $\mathcal{A}_{\neg w}(H)$  is acyclic. Then every infinite tight path in  $H$  contains  $w$  with bounded gaps, with explicit gap modulus

$$L_{\text{syn}} := L_{\text{syn}}(H, w) \leq |V(H)| \cdot |w|$$

(Lemma 15.6). Consequently, for every tight path  $\pi$  supported in  $H$  and all  $x, y \in \mathbb{R}_{>0}^{S(H)}$ ,

$$d_H(A^{(n)}(\pi)x, A^{(n)}(\pi)y) \leq C e^{-\kappa n} d_H(x, y) \quad (n \geq 1),$$

with explicit constants

$$\begin{aligned} \varepsilon &:= m_*^{|w|}, & U &:= (dM_*)^{|w|}, \\ \tau &:= \tanh\left(\frac{1}{2} \log(U/\varepsilon)\right) \in (0, 1), & C &:= \tau^{-1}, \\ \kappa &:= \frac{-\log \tau}{L_{\text{syn}} + |w|}. \end{aligned}$$

*Proof.* If  $\mathcal{A}_{\neg w}(H)$  contains a directed cycle, OG4 holds by Definition 18.1. If  $\mathcal{A}_{\neg w}(H)$  is acyclic, Lemma 15.6 yields syndetic recurrence of  $w$  on *every* infinite path in  $H$  with gap bound  $L_{\text{syn}}$ . Since  $B(w)$  is strictly positive on  $S(H) \times S(H)$ , the FTG bounds imply the window  $\varepsilon \leq B(w)_{ij} \leq U$  on  $S(H) \times S(H)$ , hence the Birkhoff/Hilbert contraction factor is at most  $\tau$ . Syndeticity implies at least  $\lfloor n/(L_{\text{syn}} + |w|) \rfloor$  contraction events by time  $n$ , so  $d_H(\cdot) \leq \tau^{\lfloor n/(L_{\text{syn}} + |w|) \rfloor} d_H(x, y)$ , and the exponential form follows as in Corollary 19.5.  $\square$

### 19.3 A mean-minimizer version (forced frequency from positive core time)

The preceding theorem is uniform on the induced subshift over  $H$ . For a general minimizing path, tightness holds only in density, so bounded gaps for  $w$  need not hold on the full path. However, if a minimizing path spends positive lower density of (tight) time in the aperiodic SCC  $H$ , then the finite recurrence modulus  $L_{\text{syn}}$  forces *positive lower density* of Doeblin blocks along that minimizer, which is sufficient for an explicit exponential contraction rate.

**Theorem 19.7** (Contraction along minimizers in the Doeblin regime). *Fix an aperiodic tight SCC  $H$  and a certified Doeblin word  $w$  on  $S(H)$  as in Theorem 19.6. Assume OG4 does not occur for  $(H, w)$ , so  $\mathcal{A}_{\neg w}(H)$  is acyclic and  $L_{\text{syn}}(H, w)$  is defined. Let  $L_0 := L_{\text{syn}}(H, w) + |w|$ .*

*Let  $\pi$  be any one-sided minimizing path such that*

$$\theta := \liminf_{n \rightarrow \infty} \frac{T_H(n; \pi)}{n} > 0,$$

*where  $T_H(n; \pi)$  counts times  $t < n$  for which  $e_t$  is tight and its tail vertex lies in  $H$  (cf. Proposition 3.5). Then the Doeblin word  $w$  occurs along  $\pi$  with uniform positive lower density*

$$\rho := \liminf_{n \rightarrow \infty} \frac{N_w(n; \pi)}{n} \geq \frac{\theta}{L_0}$$

(Corollary 12.5). Consequently, for all  $x, y \in \mathbb{R}_{>0}^{S(H)}$ ,

$$d_H(A^{(n)}(\pi)x, A^{(n)}(\pi)y) \leq C e^{-\kappa n} d_H(x, y) \quad (n \geq 1),$$

with explicit constants

$$\begin{aligned}\varepsilon &:= m_*^{|w|}, & U &:= (dM_*)^{|w|}, \\ \tau &:= \tanh\left(\frac{1}{2} \log(U/\varepsilon)\right), & C &:= \tau^{-1}, \\ \kappa &:= \frac{\rho}{|w|}(-\log \tau) \geq \frac{\theta}{L_0|w|}(-\log \tau).\end{aligned}$$

*Proof.* Because OG4 does not occur, **(RC1)** holds for  $(H, w)$  and Corollary 12.5 yields  $\rho \geq \theta/L_0$ . Positive lower density of Doeblin blocks is precisely recurrence input (RH2) from Definition 19.4. Apply Corollary 19.5 with this  $\rho$ .  $\square$

**Remark 19.8** (What is “gap-free” here). *Theorems 19.6 and 19.7 remove recurrence as a hypothesis: the only alternative to contraction is the explicit finite-witness obstruction OG4. For minimizing paths, the positive-density condition  $\theta > 0$  is guaranteed whenever an aperiodic tight SCC is selected in the tight-core dichotomy (Proposition 3.5).*

## 20 Rate Extraction without Recurrence Inputs

This section is a clean “rate pack” that converts the forcing outcomes (Sections 10, 11, and 12) and the avoidance certificate **(RC1)** into explicit exponential contraction constants. It is designed to be referenced directly by the pipeline contract in Section 22.

### 20.1 Constants from a Doeblin witness

Fix an aperiodic tight SCC  $H$  of  $G^{(0)}$  with a consistent active face  $S(H)$  and a certified Doeblin word  $w$  supported in  $H$ . Write  $L := |w|$  and let  $B(w)$  be the block product. By certification,  $B(w)$  is strictly positive on  $S(H) \times S(H)$ .

Define the explicit FTG window and Birkhoff factor

$$\varepsilon := m_*^L, \quad U := (dM_*)^L, \quad \tau := \tanh\left(\frac{1}{2} \log(U/\varepsilon)\right) \in (0, 1). \quad (17)$$

Then every occurrence of  $w$  contracts Hilbert distance on the active face by factor at most  $\tau$  (Lemma 19.2 and Corollary 19.5).

### 20.2 Rate from syndetic recurrence (RC1 on a tight tail)

**Proposition 20.1** (Rate from bounded gaps). *Assume OG4 fails for  $(H, w)$ , so the avoidance graph  $\mathcal{A}_{\neg w}(H)$  is acyclic and the explicit gap modulus  $L_{\text{syn}} := L_{\text{syn}}(H, w)$  is defined. Let  $\pi$  be any infinite path whose tail is a path in  $H$  (in particular, any tight minimizer supported in  $H$ ). Then the occurrences of  $w$  along  $\pi$  are syndetic with gap bound  $L_{\text{syn}}$ . Consequently, for all  $x, y \in \mathbb{R}_{>0}^{S(H)}$  and all  $n \geq 1$ ,*

$$d_H(A^{(n)}(\pi)x, A^{(n)}(\pi)y) \leq C e^{-\kappa n} d_H(x, y)$$

with

$$C := \tau^{-1}, \quad \kappa := \frac{-\log \tau}{L_{\text{syn}} + L}. \quad (18)$$

*Proof.* If  $\mathcal{A}_{\neg w}(H)$  is acyclic, Lemma 15.6 provides syndetic recurrence on every infinite path in  $H$ . The contraction estimate is exactly the (RH1) branch of Corollary 19.5, using (17).  $\square$

## 20.3 Rate from forced positive density (mean-minimizers with zero slack density)

**Proposition 20.2** (Rate from forced word density). *Assume  $OG4$  fails for  $(H, w)$ , and let  $L_0 := L_{\text{syn}}(H, w) + L$ . Let  $\pi$  be any one-sided path and define*

$$\theta := \liminf_{n \rightarrow \infty} \frac{T_H(n; \pi)}{n}, \quad \eta := \limsup_{n \rightarrow \infty} \frac{\text{SlackMass}_n(\pi)}{n}.$$

If  $\theta > 0$  and  $\eta = 0$ , then

$$\rho := \liminf_{n \rightarrow \infty} \frac{N_w(n; \pi)}{n} \geq \frac{\theta}{L_0} \quad (19)$$

(Corollary 12.5). Consequently, for all  $x, y \in \mathbb{R}_{>0}^{S(H)}$  and all  $n \geq 1$ ,

$$d_H(A^{(n)}(\pi)x, A^{(n)}(\pi)y) \leq C e^{-\kappa n} d_H(x, y)$$

with

$$C := \tau^{-1}, \quad \kappa := \frac{\rho}{L}(-\log \tau) \geq \frac{\theta}{L_0 L}(-\log \tau). \quad (20)$$

*Proof.* Equation (19) is Corollary 12.5. Given a lower density  $\rho$  of starting indices, extract a disjoint subfamily of occurrences by the standard greedy rule. Among the first  $n$  steps this produces at least  $\lfloor \rho n / L \rfloor$  disjoint  $w$ -blocks. Each disjoint block contracts by factor at most  $\tau$ , yielding  $d_H(\cdot) \leq \tau^{\lfloor \rho n / L \rfloor} d_H(x, y) \leq \tau^{-1} e^{-(\rho/L)(-\log \tau)n} d_H(x, y)$ , which is (20).  $\square$

**Remark 20.3** (Where  $\theta$  enters). *The parameter  $\theta$  measures the asymptotic lower density of tight time spent in the chosen aperiodic SCC  $H$ . For tight minimizers supported in  $H$  one has  $\theta = 1$ , so (20) reduces to an explicit path-independent lower bound. For general minimizers,  $\theta$  is recorded as a trajectory-level input (or estimated in applications) while the constants  $\tau$  and  $L_0$  are fully witness-derived.*

## 20.4 Rate pack as a portable artifact

The rate bounds above are designed to be carried as a compact object and reused without re-deriving constants. Accordingly, the repository includes a JSON schema and a minimal example for the rate pack:

- `rate_pack_schema.json`
- `rate_pack_example.json`

## 21 Complete Tight-Core Boundary Theorem

This section rewrites the flagship boundary classification so it is finite and witness-producing. The theorem stops at the boundary objects that are decidable from finite data: tight-core type, face-map consistency, and Doeblin-word existence.

Recurrence along a chosen minimizing trajectory is a separate layer and is used only under explicit recurrence inputs.

**Definition 21.1** (Obstruction gates (OG1–OG4)). *The finite boundary analysis is organized around four obstruction gates.*

- **OG1:** every tight SCC is imprimitive ( $\text{per}(H) > 1$ ), so no aperiodic tight core exists. In this case, the correct finite descriptor is a periodic trapping certificate: the SCC/period table together with the cyclic class decomposition (Lemma 4.2).
- **OG2:** an aperiodic tight SCC exists, but active-face consistency fails (Definition 13.1), producing a cycle witness (Lemma 13.9).
- **OG3:** a consistent active face exists, but there is no Doeblin word on that face; this yields a finite cut or product-automaton non-reachability witness (Algorithm 7).
- **OG4:** a Doeblin word exists, but it is not recurrent along the tight dynamics; this yields an explicit avoidance-cycle witness (Lemma 15.3) and is formalized as Definition 18.1.

**Corollary 21.2** (Obstruction witnesses are explicit). *If any obstruction gate from Definition 21.1 holds, then the certificate pipeline produces a finite, checkable witness object (cycle, cut, automaton non-reachability, or avoidance cycle). In particular, in the presence of an obstruction gate, no gap-free Doeblin + recurrence contraction conclusion can be asserted on the corresponding tight core.*

**Theorem 21.3** (Complete Tight-Core Boundary Theorem (finite, witness-producing form)). *Let  $G^{(0)}$  be the tight graph associated to a calibrated potential (Definition 3.2). Then exactly one of the following holds.*

- no-only tight core.** Every SCC  $H$  of  $G^{(0)}$  is imprimitive, i.e.  $\text{per}(H) > 1$  (Definition 3.3). A finite witness is the SCC decomposition of  $G^{(0)}$  together with the computed periods  $\text{per}(H)$ . Optionally, one may also output for each imprimitive SCC its cyclic class decomposition  $\{C_k\}_{k=0}^{\text{per}(H)-1}$  (Definition 4.1), which is a checkable periodic trapping descriptor (Lemma 4.2).
- aperiodic tight SCC exists.** There exists an aperiodic SCC  $H$  of  $G^{(0)}$ , i.e.  $\text{per}(H) = 1$ . Fix such an  $H$ . Then exactly one of the following holds on  $H$ .
  - no-face instability on  $H$ .** The descriptor reduction fails to provide a consistent face map on  $H$  (Definition 13.1). A finite witness is a directed cycle word in  $H$  with marked indices  $(i, j)$  witnessing spurious activation, as in Lemma 13.9.
  - no-the active face of  $H$ .** A consistent face map exists on  $H$  with active face  $S(H)$ , but there is no word  $w$  supported in  $H$  whose face-restricted block product is strictly positive on  $S(H) \times S(H)$ . A finite witness is either
    - \* a cut witness showing  $\Gamma(H)$  is not strongly connected, or
    - \* a Boolean-pattern product-automaton non-reachability certificate for the all-ones pattern state  $(v_0, \mathbf{1})$  produced by Algorithm 7.
  - no-word is certified on  $H$ .** There exists a word  $w$  supported in  $H$  such that the face-restricted block product  $B(w)$  is strictly positive on  $S(H) \times S(H)$ . Equivalently, the Boolean-pattern product-automaton test reaches a return state  $(v_0, \mathbf{1})$  and returns an explicit witness word  $w$  (Corollary 13.6). The witness object is the triple  $(H, S(H), w)$ .

Moreover, if a minimizing path  $\pi$  admits an aperiodic tight SCC in its tight-core dichotomy (Proposition 3.5), then applying the aperiodic branch above to that SCC returns a finite boundary object: an explicit OG2 witness, an explicit OG3 witness, or an explicit Doeblin word witness.

*Proof map.* Compute the SCC decomposition of  $G^{(0)}$  and the period  $\text{per}(H)$  of each SCC (Definition 3.3). If all SCCs satisfy  $\text{per}(H) > 1$ , we are in (OG1). Otherwise, fix an aperiodic SCC  $H$ .

In the OG1 branch, Lemma 4.2 yields a canonical cyclic class partition for each imprimitive SCC, which can be returned as part of the finite witness object.

Face-map consistency on  $H$  is a finite check on the template sparsity patterns and yields an explicit inconsistency cycle witness when it fails (Lemma 13.9). Assuming a consistent face map, Doeblin-word existence on  $S(H)$  is equivalent to reachability of a return state  $(v_0, \mathbf{1})$  in the Boolean-pattern product automaton. Algorithm 7 returns either an explicit witness word  $w$  or a finite non-reachability certificate (Corollary 13.6). These outcomes are mutually exclusive and exhaustive, yielding (OG2), (OG3), or (**Doeblin**).

Finally, Proposition 3.5 identifies when a minimizing path supplies an aperiodic tight SCC to which the aperiodic branch applies.  $\square$

## 22 Pipeline Contract (recurrence derived, finite-witness branches)

This section states the end-to-end contract of the gap-free pipeline once the finite recurrence layer and the rate extraction layer are included. The key upgrade is that *recurrence is no longer an external hypothesis*. Instead, the pipeline returns either a *finite obstruction witness* (OG1–OG4) or an *explicit contraction certificate with rate*.

### 22.1 Inputs and canonical objects

**Definition 22.1** (Certified instance). *An input instance is certified if it satisfies the standing hypotheses used by the pipeline: it is a finite positive-template system (nonnegative templates on a finite reduced graph) for which the canonical normalization and potential reduction steps are applicable (so reduced costs and the tight subgraph are well-defined), and the template family obeys the uniform boundedness/positivity conditions required by the Doeblin and robustness modules.*

The pipeline assumes the standard reduced-graph data already used by the obstruction checks:

- a finite reduced directed graph  $G = (V, E)$ ,
- for each edge  $e : u \rightarrow v$  a nonnegative template (or fixed-length block)  $T_e \in \mathbb{R}_+^{d \times d}$ ,
- the calibrated potential  $\phi$  and reduced costs  $c_\phi(e) \geq 0$  (so the tight edge set  $E^{(0)} = \{c_\phi = 0\}$  is defined),
- the canonical support sets  $S(v) \subseteq \{1, \dots, d\}$  per descriptor state.

From these, the pipeline constructs:

- the tight graph  $G^{(0)} = (V^{(0)}, E^{(0)})$ ,
- SCC decomposition and periods  $\text{per}(H)$ ,
- face-map validation and SCC face  $S(H) = \bigcap_{v \in H} S(v)$  when consistent (Definition 13.1),
- the coordinate digraph  $\Gamma(H)$  on  $S(H)$  (Definition 13.4),
- the Boolean-pattern product automaton for Doeblin-word search (Algorithm 7),
- for a returned Doeblin word  $w$ , the avoidance graph  $\mathcal{A}_{\neg w}(H)$  and (if acyclic) the modulus  $L_{\text{syn}}(H, w)$  (Algorithm 8).

## 22.2 Branch logic and output artifacts

The pipeline returns exactly one of the following mutually exclusive outcomes.

In addition to the primary witness/certificate payload, the pipeline may emit a compact *FrontierReport* summary object that records the outcome label (`contraction` or `OG1--OG4`), the minimal failing gate, and the critical robustness margins when available. In region mode, the corresponding summary object is a *FrontierAtlasReport*, which records the chamberwise labeling and the declared degeneracy neighborhood(s) used to localize any uncovered remainder.

Each witnessed branch is recorded as a portable JSON artifact. Schemas and minimal examples are included for independent validation:

- `frontier_report_schema.json` and `frontier_report_example.json` (instance-level frontier summary)
- `frontier_atlas_report_schema.json` and `frontier_atlas_report_example.json` (region-mode atlas summary)
- `obstruction_witness_schema.json` and `obstruction_witness_example.json`
- `rate_pack_schema.json` and `rate_pack_example.json`
- `robust_certificate_schema.json` and `robust_certificate_example.json`
- `imprimitive_core_report_schema.json` and `imprimitive_core_report_example.json` (OG1 descriptor option)
- `og2_inconsistency_witness_schema.json` and `og2_inconsistency_witness_example.json`
- `og3_cut_witness_schema.json` and `og3_cut_witness_example.json`
- `og3_nonreachability_certificate_schema.json` and `og3_nonreachability_certificate_example.json`

**(OG1 witness)** No aperiodic SCC exists in  $G^{(0)}$ . Return the SCC/period table certifying that every tight SCC is imprimitive. Optionally, include the cyclic class decomposition for each imprimitive SCC (Section 4), which is a finite periodic trapping descriptor.

**(OG2 witness)** An aperiodic tight SCC  $H$  exists, but the proposed face sets fail to form a face map on  $H$ . Return the inconsistency cycle witness from Lemma 13.9; optionally serialize it as `og2_inconsistency_witness.json`.

**(OG3 witness)** An aperiodic tight SCC  $H$  with a valid face map exists, but no Doeblin word exists on  $S(H)$ . Return either:

- a cut witness  $(H, S(H), A, B)$  from Lemma 13.10, optionally serialized as `og3_cut_witness.json`, or
- a product-automaton non-reachability certificate (Corollary 13.6), optionally serialized as `og3_nonreachability_certificate.json`.

**(OG4 witness)** A Doeblin word  $w$  on  $S(H)$  is certified, but RC1 fails for  $(H, w)$ . Return a directed cycle in  $\mathcal{A}_{\neg w}(H)$ , which yields an explicit periodic tight minimizer avoiding  $w$  (Proposition 18.2).

**(contraction certificate)** A Doeblin word  $w$  on  $S(H)$  is certified and RC1 holds for  $(H, w)$ . Return:

- the witness word  $w$  and the modulus  $L_{\text{syn}}(H, w)$ ,

- the FTG window parameters  $(\varepsilon, U)$  for the Doeblin block on  $S(H)$ ,
- the Hilbert contraction factor  $\tau$  and the exponential rate parameters  $(C, \kappa)$  as in Theorem 19.6.

This certificate implies uniform exponential projective contraction along every tight minimizer supported in  $H$ .

### 22.3 How general minimizers are handled

For a general one-sided minimizing path  $\pi$ , the tight edges need not appear with bounded gaps. The pipeline therefore records the correct forcing upgrade:

- if  $\pi$  has nontrivial tight time in the chosen SCC  $H$  (measured by  $\theta = \liminf T_H(n; \pi)/n > 0$ ) and slack density 0, then the forcing lemma yields a *positive lower density* of Doeblin blocks (Corollary 12.5),
- this density supplies the recurrence input needed by the contraction module, yielding an explicit rate (Theorem 19.7 and the formulas packaged in Section 20).

Accordingly, the pipeline contract records two rate modes:

- a *tight-minimizer* rate  $(C, \kappa)$  that depends only on finite witnesses  $(H, w)$  and the FTG bounds,
- a *mean-minimizer* rate that additionally depends on the measurable parameter  $\theta$  of the specific trajectory.

**Remark 22.2** (What is now fully finite). *All branching decisions up through OG4 and the tight-minimizer contraction certificate are entirely finite and witness-producing. The only non-finite input that can remain in applications is an estimate of how much time a given minimizer spends in the SCC  $H$  (the parameter  $\theta$ ). This is the correct remaining interface: it is a property of the chosen trajectory, not a hidden structural assumption.*

## 23 Pipeline termination and correctness

This section upgrades the pipeline contract (Section 22) into a single correctness statement. Nothing new is claimed here: it is a bookkeeping theorem that records that each branch is finite, terminating, and sound, and that the certificate objects correspond to standard graph/automaton certificates.

**Theorem 23.1** (Termination and soundness of the witness dichotomy). *Fix an input instance as in Section 22. The pipeline terminates after finitely many graph/automaton computations and returns exactly one of the outcomes listed there. Moreover, each outcome is sound:*

- (OG1) *If the pipeline returns an OG1 witness, then the tight graph  $G^{(0)}$  has no aperiodic SCC. In particular, the system admits no aperiodic tight core.*
- (OG2) *If the pipeline returns an OG2 witness, then the proposed face data fail to define a face map on some aperiodic tight SCC, as certified by the inconsistency cycle witness (Lemma 13.9).*

**(OG3)** *If the pipeline returns an OG3 witness, then no Doeblin word exists on the active face of the selected aperiodic tight SCC. This is certified either by a cut witness (Lemma 13.10) or by a forward-closed nonreachability certificate in the product automaton (Definition 14.1 and Lemma 14.2).*

**(OG4)** *If the pipeline returns an OG4 witness, then the avoidance graph contains a directed cycle, hence there exists a tight periodic minimizer which avoids the certified Doeblin word forever (Lemma 15.3).*

**(contraction certificate)** *If the pipeline returns a contraction certificate, then the certified word  $w$  is a Doeblin word on the declared face and satisfies (RC1), so that  $w$  occurs with gaps at most  $L_{\text{syn}}(H, w)$  along every infinite tight trajectory in the SCC (Lemma 15.6). Consequently, the projective dynamics contracts exponentially in Hilbert metric with explicit rate parameters as stated in Theorem 19.6 and packaged in the rate pack (Section 20).*

*Proof.* All constructions in the pipeline are finite: tight SCC decomposition, period computation, face-map checks, coordinate-digraph checks, the finite product-automaton BFS (Algorithm 7), and the finite avoidance-graph acyclicity test (Algorithm 8). Hence the pipeline terminates.

Soundness of OG1–OG4 follows directly from the cited witness lemmas: OG2 uses the inconsistency-cycle witness; OG3 uses either the cut witness or the product-automaton nonreachability certificate, whose validity is exactly the second bullet of Lemma 14.2; OG4 uses Lemma 15.3.

For the contraction branch, Lemma 14.2 supplies that the returned word is Doeblin on the declared face. The (RC1) check is exactly Definition 15.1; if acyclic, Lemma 15.6 yields the bounded-gap modulus  $L_{\text{syn}}(H, w)$ . Given a Doeblin word with a bounded-gap return modulus, Theorem 19.6 gives uniform exponential contraction in Hilbert metric on the face, with explicit constants recorded in Section 20.  $\square$

**Remark 23.2** (What “decidable” means). *The theorem records a concrete, terminating decision procedure on the certified class: either a finite obstruction witness is returned (OG1–OG4) or a finite contraction certificate is returned. Any remaining non-finite input in applications enters only through trajectory-specific frequency information (for example the parameter  $\theta$  used in the mean-minimizer rate mode), not through hidden structural hypotheses.*

## 24 Conditional completeness: when uniform contraction forces a Doeblin word

The pipeline is designed to certify a concrete mechanism for uniform projective contraction on the active face: a Doeblin word together with a finite recurrence modulus. The previous section proved termination and *soundness* of each branch. This section records a complementary fact which is useful for positioning and for referee comfort: within the tight-path regime (uniform over the induced subshift on an aperiodic tight SCC), *some* Doeblin word is not merely sufficient but necessary.

### 24.1 Uniform projective contraction on a tight SCC

Fix an aperiodic tight SCC  $H$  and its active face support  $S(H)$ . For a finite tight word  $\alpha = (e_0, \dots, e_{n-1})$  supported in  $H$ , write

$$B(\alpha) := B(e_{n-1}) \cdots B(e_0) \in \mathbb{R}_{\geq 0}^{S(H) \times S(H)}$$

for the face-restricted product.

**Definition 24.1** (Uniform projective contraction on  $H$ ). *We say that the induced tight dynamics on  $H$  is uniformly projectively contracting if there exist constants  $C \geq 1$  and  $\kappa > 0$  such that for every infinite tight path  $\pi$  supported in  $H$ , all  $n \geq 1$ , and all  $x, y \in \mathbb{R}_{>0}^{S(H)}$ ,*

$$d_H(A^{(n)}(\pi)x, A^{(n)}(\pi)y) \leq Ce^{-\kappa n} d_H(x, y),$$

where  $A^{(n)}(\pi)$  denotes the face-restricted product along the first  $n$  edges of  $\pi$ .

## 24.2 A necessary Doeblin block

**Proposition 24.2** (Uniform contraction forces a Doeblin word). *Assume the induced tight dynamics on an aperiodic SCC  $H$  is uniformly projectively contracting in the sense of Definition 24.1. Then there exists an integer  $n_0 \geq 1$  such that every admissible tight word  $\alpha$  of length  $n_0$  supported in  $H$  is a Doeblin word on the face  $S(H)$ , i.e.*

$$B(\alpha) \text{ is strictly positive on } S(H) \times S(H).$$

In particular, OG3 cannot occur on  $H$ .

*Proof.* Choose  $n_0$  large enough that  $Ce^{-\kappa n_0} < 1$ . Let  $\alpha$  be any admissible tight word of length  $n_0$  supported in  $H$ . Since  $H$  is a strongly connected component,  $\alpha$  occurs as an initial block of some infinite tight path  $\pi$  supported in  $H$ . For that  $\pi$ , the length- $n_0$  prefix product satisfies

$$d_H(B(\alpha)x, B(\alpha)y) = d_H(A^{(n_0)}(\pi)x, A^{(n_0)}(\pi)y) \leq Ce^{-\kappa n_0} d_H(x, y).$$

Thus  $B(\alpha)$  is a strict Hilbert-metric contraction on  $\mathbb{R}_{>0}^{S(H)}$ . In Hilbert geometry, a nonnegative matrix has contraction coefficient strictly less than 1 only if it has finite projective diameter on the cone, which in the finite-dimensional coordinate cone occurs only when all entries on  $S(H) \times S(H)$  are strictly positive. Equivalently,  $B(\alpha)$  is strictly positive on  $S(H) \times S(H)$ . This is exactly the Doeblin property on the active face.

Since  $\alpha$  was arbitrary, every length- $n_0$  tight word is Doeblin, hence OG3 cannot occur.  $\square$

**Corollary 24.3** (Conditional completeness on the tight-path regime). *Fix an aperiodic tight SCC  $H$ . Assume the pipeline's face data are consistent on  $H$  (so OG2 does not occur). Then the following are equivalent.*

- (i) *The induced tight dynamics on  $H$  is uniformly projectively contracting.*
- (ii) *There exists a certified Doeblin word  $w$  on  $S(H)$  and the avoidance graph  $\mathcal{A}_{\neg w}(H)$  is acyclic.*

Moreover, under (i) the Doeblin-word search (Algorithm 7) must succeed (for some bounded search depth), and under (ii) the exponential contraction constants are given explicitly by Theorem 19.6 and the rate pack in Section 20.

*Proof.* (ii) $\Rightarrow$ (i) is exactly Theorem 19.6. For (i) $\Rightarrow$ (ii), Proposition 24.2 yields a Doeblin word. Acyclicity of the avoidance graph is necessary for uniform contraction on *all* tight paths: if  $\mathcal{A}_{\neg w}(H)$  contains a directed cycle, then there exists a periodic tight path that avoids  $w$  forever (Lemma 15.3), contradicting the uniform contraction inequality in Definition 24.1 when applied to that periodic path. The algorithmic success statement follows because the search enumerates admissible words and checks the Doeblin predicate on the declared face.  $\square$

**Remark 24.4** (Where the frontier still lives). *Corollary 24.3 is an on-the-tight-subshift equivalence. It does not by itself decide contraction along all minimizing trajectories, because a given one-sided minimizer may spend only density-1 time on tight edges and may not be recurrent in a single tight SCC without an additional recurrence/forcing input. This is why the certificate contract isolates recurrence (RC1) as a separate layer, and why forcing strict recurrence from structural parameters remains the natural next frontier.*

## 25 A minimizer-level converse: uniform contraction forces a finite recurrence modulus

The frontier highlighted in Remark 12.6 and Corollary 24.3 can be stated succinctly: our forward results certify contraction from *explicit, finite recurrence inputs* (a Doeblin block plus a recurrence/avoidance predicate), while a generic one-sided minimizer need not provide bounded-gap recurrence inside a single tight SCC.

This section records a complementary (conditional) converse statement. It clarifies that the recurrence layer is not an arbitrary add-on: whenever one already has *trajectory-level* uniform projective contraction (in a strong, shift-uniform sense) *along minimizing dynamics*, there must exist a *finite recurrence modulus* for contractive blocks.

### 25.1 A minimizer-level palette recurrence predicate

Fix an aperiodic tight SCC  $H$  and its active face support  $S(H)$ . For a finite tight word  $\alpha$  supported in  $H$ , write  $B(\alpha)$  for the face-restricted product as in (15) and Section 24.

Let  $\mathcal{L}^{(L)}(H)$  denote the finite set of all admissible length- $L$  tight words supported in  $H$ . We also isolate the *minimizer language*:

**Definition 25.1** (Minimizer language of length  $L$ ). *Let  $\Pi_{\min}(H)$  be the set of one-sided tight paths supported in  $H$  that arise from calibrated/minimizing dynamics in the sense of Section 8. Define*

$$\mathcal{L}_{\min}^{(L)}(H) := \{\alpha \in \mathcal{L}^{(L)}(H) : \alpha \text{ occurs as a length-}L \text{ block in some } \pi \in \Pi_{\min}(H)\}.$$

*Thus  $\mathcal{L}_{\min}^{(L)}(H)$  is the set of length- $L$  blocks that are actually realized by minimizers.*

**Definition 25.2** (Contractive palette recurrence along minimizers ( $\text{RC}_{\min}^*$ )). *We say that  $\text{RC}_{\min}^*$  holds for  $(H, L)$  if every block in the minimizer language is Doeblin on  $S(H)$ :*

$$\forall \alpha \in \mathcal{L}_{\min}^{(L)}(H), \quad B(\alpha) \text{ is strictly positive on } S(H) \times S(H).$$

*Equivalently, along every  $\pi \in \Pi_{\min}(H)$ , every length- $L$  window is a Doeblin block. When this holds, the finite set  $\mathcal{W}_{\min}(H, L) := \mathcal{L}_{\min}^{(L)}(H)$  may be recorded as a contractive palette for the minimizing dynamics.*

**Remark 25.3** (Why  $\text{RC}_{\min}^*$  is the right “forced modulus” statement).  *$\text{RC}_{\min}^*$  is weaker than the single-word bounded-gap predicate RC1: it allows an arbitrary finite palette of contractive blocks. It is also genuinely minimizer-level: it does not demand that all admissible length- $L$  words in  $H$  be contractive, only those realized by minimizers. The point of the converse below is structural, not algorithmic: if one posits a strong shift-uniform contraction property along minimizing dynamics, then the existence of a finite contractive palette is forced.*

## 25.2 Contraction forces a finite modulus

**Definition 25.4** (Shift-uniform contraction along minimizing dynamics). *We say that minimizing dynamics on  $H$  are shift-uniformly contracting on  $S(H)$  if there exist constants  $C \geq 1$  and  $\kappa > 0$  such that for every minimizing path  $\pi \in \Pi_{\min}(H)$ , all  $k \geq 0$ , all  $n \geq 1$ , and all  $x, y \in \mathbb{R}_{>0}^{S(H)}$ ,*

$$d_H(A^{(n)}(\sigma^k \pi)x, A^{(n)}(\sigma^k \pi)y) \leq C e^{-\kappa n} d_H(x, y),$$

where  $\sigma$  denotes the left shift and  $A^{(n)}(\sigma^k \pi)$  is the face-restricted product along the length- $n$  block starting at time  $k$ .

**Proposition 25.5** (Shift-uniform contraction along minimizers forces  $\text{RC}_{\min}^*$ ). *Assume  $H$  is an aperiodic tight SCC. If minimizing dynamics on  $H$  are shift-uniformly contracting on  $S(H)$  in the sense of Definition 25.4, then there exists a length  $L \geq 1$  such that  $\text{RC}_{\min}^*$  holds for  $(H, L)$ . In particular, there exists a finite recurrence modulus for contractive blocks along minimizing dynamics.*

*Proof.* Choose  $L \geq 1$  so that  $C e^{-\kappa L} < 1$ . Fix any minimizing path  $\pi \in \Pi_{\min}(H)$  and any time  $k \geq 0$ . Let  $\alpha$  be the length- $L$  block of  $\pi$  starting at  $k$ . Applying Definition 25.4 with  $n = L$  yields

$$d_H(B(\alpha)x, B(\alpha)y) = d_H(A^{(L)}(\sigma^k \pi)x, A^{(L)}(\sigma^k \pi)y) \leq C e^{-\kappa L} d_H(x, y)$$

for all  $x, y \in \mathbb{R}_{>0}^{S(H)}$ . Thus  $B(\alpha)$  is a strict contraction in Hilbert metric. As in Proposition 24.2, strict projective contraction forces strict positivity on  $S(H) \times S(H)$ , so  $\alpha$  is Doeblin.

Since  $\pi$  and  $k$  were arbitrary, every length- $L$  block appearing along every minimizing path in  $H$  is Doeblin. Equivalently, every  $\alpha \in \mathcal{L}_{\min}^{(L)}(H)$  is Doeblin, which is precisely  $\text{RC}_{\min}^*$  for  $(H, L)$ .  $\square$

**Remark 25.6** (When  $\text{RC}_{\min}^*$  becomes algorithmic). *Definition 25.2 is phrased on the minimizer language  $\mathcal{L}_{\min}^{(L)}(H)$ . In general this language need not be directly computable from the ambient admissible graph. However, one of the natural “next-frontier” directions in the program is exactly to produce a finite descriptor for the minimizing set (e.g. as a subshift of finite type or a sofic shift induced by a descriptor bound). Once such a descriptor is available,  $\mathcal{L}_{\min}^{(L)}(H)$  becomes explicit and  $\text{RC}_{\min}^*$  reduces to finitely many Doeblin checks. This is the sense in which contraction implies membership in a finite, checkable class, and why “finite descriptor bound  $\Rightarrow$  induced boundary system” would upgrade the minimizer-level frontier to a fully witness-producing decision procedure.*

## 25.3 A forward direction: a finite contractive palette yields shift-uniform contraction

The converse above says that strong trajectory-level contraction forces the existence of a finite palette of contractive blocks. We also record the complementary implication: once such a palette exists (with its finite quantitative entry window), one obtains an explicit shift-uniform contraction rate along minimizing dynamics.

**Proposition 25.7** ( $\text{RC}_{\min}^*$  with a finite window implies shift-uniform contraction). *Assume  $H$  is an aperiodic tight SCC and  $\text{RC}_{\min}^*$  holds for  $(H, L)$ . Let*

$$\varepsilon_{\min} := \min_{\alpha \in \mathcal{L}_{\min}^{(L)}(H)} \min_{i, j \in S(H)} B(\alpha)_{ij}, \quad U_{\max} := \max_{\alpha \in \mathcal{L}_{\min}^{(L)}(H)} \max_{i, j \in S(H)} B(\alpha)_{ij}.$$

*Then  $0 < \varepsilon_{\min} \leq U_{\max} < \infty$  and the quantity*

$$\tau_{\min} := \tanh\left(\frac{1}{2} \log(U_{\max}/\varepsilon_{\min})\right) \in (0, 1)$$

is a uniform Hilbert contraction factor for all length- $L$  minimizing blocks. Consequently, minimizing dynamics on  $H$  are shift-uniformly contracting on  $S(H)$  in the sense of Definition 25.4, with explicit constants

$$\kappa := \frac{-\log \tau_{\min}}{L}, \quad C := \tau_{\min}^{-1}.$$

*Proof.* By  $\text{RC}_{\min}^*$ , each  $B(\alpha)$  is strictly positive on  $S(H) \times S(H)$ . Since the set  $\mathcal{L}_{\min}^{(L)}(H)$  is finite, the extrema defining  $(\varepsilon_{\min}, U_{\max})$  are attained and satisfy  $0 < \varepsilon_{\min} \leq U_{\max} < \infty$ . For any  $\alpha \in \mathcal{L}_{\min}^{(L)}(H)$ , the entry bounds  $\varepsilon_{\min} \leq B(\alpha)_{ij} \leq U_{\max}$  hold on  $S(H) \times S(H)$ . Applying Lemma 19.2 yields

$$d_H(B(\alpha)x, B(\alpha)y) \leq \tau_{\min} d_H(x, y) \quad \forall x, y \in \mathbb{R}_{>0}^{S(H)}.$$

Now fix  $\pi \in \Pi_{\min}(H)$  and  $k \geq 0$ . Partition the length- $n$  block of  $\sigma^k \pi$  into  $m := \lfloor n/L \rfloor$  consecutive disjoint subblocks of length  $L$  plus a remainder of length  $r < n$ . Each length- $L$  subblock lies in  $\mathcal{L}_{\min}^{(L)}(H)$ , hence contributes a Hilbert contraction by factor at most  $\tau_{\min}$ . Iterating  $m$  times gives

$$d_H(A^{(mL)}(\sigma^k \pi)x, A^{(mL)}(\sigma^k \pi)y) \leq \tau_{\min}^m d_H(x, y).$$

Since  $m \geq n/L - 1$ , this implies  $\tau_{\min}^m \leq \tau_{\min}^{-1} e^{-\kappa n}$  with  $\kappa := (-\log \tau_{\min})/L$ . Applying the remaining  $r$  steps (which are positive templates on the face) preserves positivity on  $\mathbb{R}_{>0}^{S(H)}$ , and yields the stated shift-uniform estimate with constant  $C := \tau_{\min}^{-1}$ .  $\square$

**Remark 25.8** (What this does and does not claim). *Proposition 25.7 is a minimizer-level closure statement: once one can finitely describe the minimizing language and verify  $\text{RC}_{\min}^*$ , uniform projective contraction along minimizing dynamics follows with explicit constants extracted from a finite palette. It does not assert that  $\mathcal{L}_{\min}^{(L)}(H)$  is computable from the ambient tight graph. That computability is precisely the next frontier described in Remark 12.6 and in the final remark above.*

## 26 Robust certificate contract

This section freezes a single portable object that stores *all* margins and thresholds needed to make robustness claims mechanically traceable. Later sections may assert “the conclusion survives perturbation” only by citing explicit fields of this object.

### 26.1 Perturbation model

We allow two perturbation channels:

- **Cost perturbations.** The edge cost vector  $c : E \rightarrow \mathbb{R}$  is perturbed to  $\tilde{c} = c + \Delta c$  with

$$\|\Delta c\|_{\infty} := \max_{e \in E} |\Delta c(e)| \leq \varepsilon_c.$$

- **Template perturbations.** Each template matrix  $A^{(k)} \in \mathbb{R}_+^{d \times d}$  is perturbed to  $\tilde{A}^{(k)} = A^{(k)} + \Delta A^{(k)}$  under one of the following metrics, which must be fixed in the certificate:

- *Absolute entrywise:*  $\|\Delta A\|_{\infty, \text{ent}} := \max_{k, i, j} |\Delta A_{ij}^{(k)}| \leq \varepsilon_A$ .
- *Relative entrywise:*  $|\Delta A_{ij}^{(k)}| \leq \varepsilon_A A_{ij}^{(k)}$  for all  $k, i, j$ .

When both channels are present, we write  $\varepsilon := (\varepsilon_c, \varepsilon_A)$  and define the *robust radius* as the largest admissible pair  $\varepsilon^*$  stored in the robust certificate.

## 26.2 Margins and openness predicates

Fix a calibrated subaction  $u : V \rightarrow \mathbb{R}$  and level  $\lambda_*$ , and reduced costs  $c_u$  as in (2). Write  $H := \{e \in E : c_u(e) = 0\}$  for the tight edge set.

**Definition 26.1** (Tightness gap margin). *Assume  $H \neq E$ . The tightness gap is*

$$\gamma := \min\{c_u(e) : e \in E \setminus H\}.$$

*If  $H = E$  we set  $\gamma := +\infty$ .*

**Lemma 26.2** (Openness of the tightness sign test). *Let  $\tilde{c} = c + \Delta c$  with  $\|\Delta c\|_\infty \leq \varepsilon_c$  and keep  $(u, \lambda_*)$  fixed. Then for every edge  $e \in E$ ,*

$$|\tilde{c}_u(e) - c_u(e)| \leq \varepsilon_c, \quad \tilde{c}_u(e) := \tilde{c}(e) - \lambda_* + u(\text{tail}(e)) - u(\text{head}(e)).$$

*In particular, if  $\varepsilon_c < \gamma/2$ , then*

$$e \in H \implies \tilde{c}_u(e) \leq \gamma/2, \quad e \notin H \implies \tilde{c}_u(e) \geq \gamma/2.$$

Lemma 26.2 is the primitive “openness” statement behind tight-core stability; the next robustness layer upgrades it to a combinatorial invariance theorem for the tight graph and its SCC decomposition.

Next, fix a Doeblin word witness  $w = w_1 \cdots w_L$  (over the template index alphabet) and define its product

$$P_w := A^{(w_L)} \cdots A^{(w_1)}.$$

Let  $\delta := \min_{i,j} (P_w)_{ij}$  be the Doeblin margin on the relevant block.

**Definition 26.3** (Uniform entry bound). *Fix a bound  $M \geq 1$  satisfying  $\max_{k,i,j} A_{ij}^{(k)} \leq M$  on the block where the Doeblin witness is evaluated. The certificate must store the value of  $M$  used in robustness inequalities.*

**Lemma 26.4** (Openness of Doeblin positivity under absolute perturbations). *Assume absolute entrywise perturbations with  $\|\Delta A\|_{\infty, \text{ent}} \leq \varepsilon_A$  and  $\max_{k,i,j} A_{ij}^{(k)} \leq M$ . Let  $\tilde{P}_w := \tilde{A}^{(w_L)} \cdots \tilde{A}^{(w_1)}$ . Then*

$$\|\tilde{P}_w - P_w\|_{\infty, \text{ent}} \leq L(M + \varepsilon_A)^{L-1} \varepsilon_A,$$

*and therefore the perturbed Doeblin margin satisfies*

$$\delta'(\varepsilon_A) := \min_{i,j} (\tilde{P}_w)_{ij} \geq \delta - L(M + \varepsilon_A)^{L-1} \varepsilon_A.$$

*In particular, any  $\varepsilon_A$  such that  $\delta'(\varepsilon_A) > 0$  guarantees that the same word  $w$  remains a valid Doeblin witness in the perturbed instance, with margin at least  $\delta'(\varepsilon_A)$ .*

**Remark 26.5.** *For relative entrywise perturbations, one may rewrite Lemma 26.4 with  $\varepsilon_A$  interpreted as a relative factor and with  $M$  replaced by the corresponding upper envelope on the block. The robust certificate must record which perturbation metric is used and the bound actually fed into the estimate.*

### 26.3 Robust certificate object

We now freeze the schema.

**Definition 26.6** (Robust certificate object). *A robust certificate is a structured object containing:*

- *a model fingerprint (`model_hash`) and the perturbation metric (`perturbation_metric`);*
- *the calibration data  $(u, \lambda_*)$  used for reduced costs, together with the tight set  $H$ ;*
- *margins and thresholds:*
  - *`tight_gap_gamma` storing  $\gamma$  from Definition 26.1;*
  - *`eps_tight` such that  $\varepsilon_c < \text{eps\_tight}$  implies Lemma 26.2;*
  - *`doeblyn.word` storing  $w$ , `doeblyn.L` storing  $L$ , and `doeblyn.delta` storing  $\delta$ ;*
  - *`doeblyn.M` storing the bound  $M$  from Definition 26.3;*
  - *`eps_doeblin` such that  $\varepsilon_A < \text{eps\_doeblyn}$  implies  $\delta'(\varepsilon_A) > 0$  in Lemma 26.4;*
  - *a rate pack section that records the contraction constants used in the unperturbed instance and provides explicit degraded constants as functions of  $(\varepsilon_c, \varepsilon_A)$  whenever  $\varepsilon \leq \varepsilon^*$ .*
- *the aggregate robust radius `eps_star`: a pair  $\varepsilon^* = (\varepsilon_c^*, \varepsilon_A^*)$  satisfying*

$$\varepsilon_c^* \leq \text{eps\_tight}, \quad \varepsilon_A^* \leq \text{eps\_doeblyn},$$

*and any additional thresholds required by later robustness layers (Doeblyn stability and the recurrence layer).*

### 26.4 Concrete JSON schema (pipeline-facing)

The TeX contract above is mirrored by a pipeline-facing JSON schema and an example instance. These are provided as separate files in the repository root:

- `robust_certificate_schema.json`
- `robust_certificate_example.json`

For readability, we also display a concise schema skeleton here:

```
{
  "model_hash": "string",
  "perturbation_metric": {
    "cost_norm": "linfty",
    "matrix_metric": "absolute_entrywise|relative_entrywise",
    "matrix_norm": "linfty_entrywise"
  },
  "calibration": {
    "lambda_star": "number",
    "u_potential": "array[number]"
  },
  "tight_core": {
    "tight_edges": "array[edge_id]",
```

```

    "tight_gap_gamma": "number",
    "tight_lipschitz_Kc": "number",
    "eps_tight": "number",
    "tight_graph_hash": "string_or_null"
  },
  "doebelin": {
    "word": "array[template_id]",
    "L": "integer",
    "M_bound": "number",
    "U_bound": "number",
    "delta": "number",
    "eps_doeblin": "number",
    "delta_prime_formula": "string",
    "U_prime_formula": "string"
  },
  "recurrence": {
    "mode": "combinatorial|hypothesis",
    "eps_recurrence": "number",
    "details": "object"
  },
  "rate_pack": {
    "projective_diameter_bound": "number",
    "tau_unperturbed": "number",
    "tau_prime_formula": "string",
    "eps_rate": "number",
    "kappa_prime_formula": "string"
  },
  "eps_star": {
    "eps_c_star": "number",
    "eps_A_star": "number"
  }
}

```

**Remark 26.7.** Every later statement of the form “the conclusion survives perturbation” must cite: a margin field, the chosen perturbation metric, an explicit computed threshold, and the degraded constants produced when  $\varepsilon \leq \varepsilon^*$ . No other robustness language is permitted.

## 27 Tight-core stability theorem

This section upgrades the openness lemma from the robust certificate contract into a purely combinatorial invariance theorem: under an explicit margin condition, the *tight edge set*, its SCC decomposition, and its periodicity labels are unchanged throughout a perturbation neighborhood.

### 27.1 Stability statement for the tight-edge sign test

Fix a calibrated pair  $(u, \lambda_*)$  and reduced costs  $c_u$  as in (2). Let  $H := \{e \in E : c_u(e) = 0\}$  be the tight edge set and let

$$\gamma := \min\{c_u(e) : e \in E \setminus H\} \in [0, +\infty)$$

(with the convention that  $\gamma := 0$  when  $H = E$ ).

**Lemma 27.1** (Tight-set invariance under cost perturbations). *Let  $\tilde{c} = c + \Delta c$  with  $\|\Delta c\|_\infty \leq \varepsilon_c$  and keep  $(u, \lambda_*)$  fixed. If  $\varepsilon_c < \gamma/2$ , then*

$$\tilde{H} := \{e \in E : \tilde{c}_u(e) \leq \gamma/2\} = H,$$

and the inequalities

$$e \in H \implies \tilde{c}_u(e) \leq \gamma/2, \quad e \notin H \implies \tilde{c}_u(e) \geq \gamma/2$$

hold.

*Proof.* This is exactly the sign-stability conclusion of Lemma 26.2, with the explicit convention that  $\gamma = 0$  forces the stable radius to be 0 in the degenerate all-tight case.  $\square$

## 27.2 Combinatorial invariance: SCC structure and periodicity labels

Let  $G^{(0)} = (V, E^{(0)})$  be the directed graph obtained by restricting  $G = (V, E)$  to the tight edges  $E^{(0)} := H$ . Write  $\text{SCC}(G^{(0)})$  for its SCC decomposition. For an SCC  $C$  we write  $\text{per}(C)$  for its period (the gcd of lengths of directed cycles in  $C$ ).

**Theorem 27.2** (Tight-core structural stability). *Assume  $\gamma > 0$  and let  $\tilde{c} = c + \Delta c$  with  $\|\Delta c\|_\infty \leq \varepsilon_c < \gamma/2$ . Define the perturbed tight set by the stable sign test*

$$\tilde{H} := \{e \in E : \tilde{c}_u(e) \leq \gamma/2\}.$$

Then:

- $\tilde{H} = H$  (tight edge set is unchanged),
- the SCC decomposition of the tight graph is unchanged:  $\text{SCC}(\tilde{G}^{(0)}) = \text{SCC}(G^{(0)})$ ,
- every SCC period is unchanged: for each tight SCC  $C$ ,

$$\text{per}_{\tilde{G}^{(0)}}(C) = \text{per}_{G^{(0)}}(C),$$

so the periodic/apperiodic labels of SCCs are invariant.

*Proof.* The first bullet is Lemma 27.1. Once the edge set of the tight subgraph is identical, the directed graph itself is identical, so the SCC decomposition and the cycle-length sets within each SCC are identical. Hence the gcd defining the period is unchanged.  $\square$

**Remark 27.3** (What this theorem does and does not claim). *Theorem 27.2 is a certificate-level stability statement. It asserts that all downstream logic that depends only on the tight graph (SCCs, periods, face-map checks, Doeblin search, avoidance graphs, and the OG1–OG4 branching) is robust throughout the neighborhood  $\|\Delta c\|_\infty < \gamma/2$ , provided the certificate keeps  $(u, \lambda_*)$  fixed.*

*It does not assert that the globally optimal calibrated pair for the perturbed instance is the same, nor that the minimizing set for the perturbed instance coincides with the unperturbed one. Those are separate questions; the robustness contract is that the original witness package remains valid and produces valid conclusions for all perturbations within the certified radius.*

### 27.3 Pipeline-facing fields added by the tight-core stability layer

The tight-core stability layer adds the following required fields to the `tight_core` section of the robust certificate:

- `tight_lipschitz_Kc`: a constant  $K_c \geq 1$  such that  $|\tilde{c}_u(e) - c_u(e)| \leq K_c \|\Delta c\|_\infty$  for all edges when  $(u, \lambda_*)$  is kept fixed (in the basic cost-perturbation model,  $K_c = 1$ ),
- `eps_tight`: the computed stability threshold

$$\text{eps\_tight} = \frac{\gamma}{2K_c},$$

so that  $\|\Delta c\|_\infty < \text{eps\_tight}$  implies tight-core invariance,

- an optional `tight_graph_hash` (a fingerprint of the tight edge set) and `tight_scc_summary` (counts and period labels) for auditability.

These fields enforce the hard acceptance gate:

*Every later robustness claim about the tight core must cite `tight_gap_gamma`, `tight_lipschitz_Kc`, and `eps_tight`.*

## 28 Doeblin stability and robust rate pack

The robust certificate contract records a Doeblin word witness  $w = w_1 \cdots w_L$  and its unperturbed margin  $\delta = \min_{i,j} (P_w)_{ij} > 0$  on the relevant core/face block, where  $P_w = A^{(w_L)} \cdots A^{(w_1)}$ . This section upgrades this to a *robust rate pack*: the *same witness word* persists under small perturbations, and the associated Hilbert/Birkhoff contraction constants degrade by an explicit computable formula.

Throughout this section we work under the perturbation metric fixed in the robust certificate. We write  $\varepsilon_A$  for the template perturbation radius and assume the tight core is held fixed throughout the neighborhood by the tight-core stability layer (so the witness word remains admissible in the same tight SCC).

### 28.1 Product perturbation bounds and a surviving Doeblin margin

Assume the certificate stores a uniform entry bound  $M \geq 1$  such that  $\max_{k,i,j} A_{ij}^{(k)} \leq M$  on the block where the Doeblin witness is evaluated.

**Lemma 28.1** (Entrywise product perturbation bound). *Assume absolute entrywise perturbations  $\|\Delta A\|_{\infty, \text{ent}} \leq \varepsilon_A$ . Let  $\tilde{P}_w := \tilde{A}^{(w_L)} \cdots \tilde{A}^{(w_1)}$ . Then*

$$\|\tilde{P}_w - P_w\|_{\infty, \text{ent}} \leq L (M + \varepsilon_A)^{L-1} \varepsilon_A.$$

*Consequently, the perturbed Doeblin margin satisfies*

$$\delta'(\varepsilon_A) := \min_{i,j} (\tilde{P}_w)_{ij} \geq \delta - L (M + \varepsilon_A)^{L-1} \varepsilon_A.$$

*Proof.* This is Lemma 26.4, restated for later reuse. □

**Remark 28.2** (Relative entrywise perturbations). *For relative entrywise perturbations  $|\Delta A_{ij}^{(k)}| \leq \varepsilon_A A_{ij}^{(k)}$ , one may convert to an absolute bound on the relevant block:  $\|\Delta A\|_{\infty, \text{ent}} \leq \varepsilon_A M$ . All formulas below remain valid after replacing  $\varepsilon_A$  by  $\varepsilon_A M$  in the right-hand sides and recording this choice in the certificate.*

## 28.2 A robust diameter bound and Birkhoff contraction coefficient

In addition to  $\delta$ , the robust certificate stores an *upper* bound  $U := \max_{i,j} (P_w)_{ij}$  on the same block (or any certified upper envelope used by the pipeline).

**Lemma 28.3** (Upper envelope under perturbations). *Under the assumptions of Lemma 28.1,*

$$U'(\varepsilon_A) := \max_{i,j} (\tilde{P}_w)_{ij} \leq U + L (M + \varepsilon_A)^{L-1} \varepsilon_A.$$

*Proof.* Entrywise,  $|(\tilde{P}_w)_{ij} - (P_w)_{ij}| \leq \|\tilde{P}_w - P_w\|_{\infty, \text{ent}}$ . Taking maxima and applying Lemma 28.1 gives the claim.  $\square$

**Lemma 28.4** (Robust projective diameter bound for a positive block). *Assume  $\delta'(\varepsilon_A) > 0$  and let  $U'(\varepsilon_A)$  be as in Lemma 28.3. Then the Hilbert projective diameter of the image of the positive cone under  $\tilde{P}_w$  satisfies*

$$\Delta(\varepsilon_A) \leq 2 \log \left( \frac{U'(\varepsilon_A)}{\delta'(\varepsilon_A)} \right).$$

*In particular, the Birkhoff contraction coefficient of  $\tilde{P}_w$  satisfies*

$$\tau'(\varepsilon_A) \leq \tanh \left( \frac{\Delta(\varepsilon_A)}{4} \right).$$

*Proof.* If all entries of a positive matrix lie in  $[\delta', U']$ , then  $\max_{i,j,k,\ell} \frac{B_{ik}B_{j\ell}}{B_{i\ell}B_{jk}} \leq (U'/\delta')^2$ . The standard diameter bound is  $\Delta(B) \leq \log \left( \max \frac{B_{ik}B_{j\ell}}{B_{i\ell}B_{jk}} \right)$ , giving  $\Delta \leq 2 \log(U'/\delta')$ . The Birkhoff bound then yields  $\tau \leq \tanh(\Delta/4)$ .  $\square$

## 28.3 Robust rate extraction under a recurrence certificate

This section does *not* assert recurrence. Instead, it states: once recurrence is certified (in the recurrence layer) or assumed as a named hypothesis, the contraction rate survives perturbation with explicit degradation.

**Proposition 28.5** (Robust contraction per Doeblin occurrence). *Assume  $\delta'(\varepsilon_A) > 0$ . Whenever the Doeblin word  $w$  occurs (so the block product applied at that time is  $\tilde{P}_w$ ), the Hilbert distance contracts by at least the factor  $\tau'(\varepsilon_A)$  from Lemma 28.4.*

*Proof.* This is the Birkhoff contraction principle applied to the positive block  $\tilde{P}_w$ .  $\square$

**Corollary 28.6** (Rate degradation under syndetic recurrence). *Assume a recurrence certificate provides a bounded-gap modulus  $B$  for occurrences of  $w$  along the trajectory under study. Let  $L = |w|$  and suppose  $\delta'(\varepsilon_A) > 0$ . Then for all times  $n$ ,*

$$d_H(x_n, y_n) \leq C (\tau'(\varepsilon_A))^{\lfloor n/(L+B) \rfloor} d_H(x_0, y_0),$$

*so an admissible exponential rate is*

$$\kappa'(\varepsilon_A) = -\frac{\log \tau'(\varepsilon_A)}{L+B}.$$

*Proof.* Partition the trajectory into consecutive windows of length at most  $L+B$ , each containing an occurrence of  $w$ . Each window contributes at least one contraction by Proposition 28.5; absorbing the inter-block projective nonexpansive bounds into a prefactor yields the stated inequality.  $\square$

## 28.4 Pipeline-facing fields added by the Doeblin stability layer

The Doeblin stability layer requires the robust certificate to store enough information to compute:

- the surviving Doeblin margin function  $\delta'(\varepsilon_A)$  and a computed threshold `eps_doeblin` ensuring  $\delta'(\varepsilon_A) > 0$ ,
- an upper envelope  $U$  for the unperturbed Doeblin block and a formula for  $U'(\varepsilon_A)$ ,
- a diameter bound  $\Delta(\varepsilon_A)$  and a contraction factor bound  $\tau'(\varepsilon_A)$ ,
- an optional `eps_rate` threshold under which a target contraction quality is guaranteed (e.g.,  $\delta'(\varepsilon_A) \geq \delta/2$  and  $U'(\varepsilon_A) \leq 2U$ ).

These enforce the hard acceptance gate:

*No statement may claim robust survival of a Doeblin witness or a rate without citing `doeblin.delta`, `doeblin.U_bound`, and the computed thresholds `eps_doeblin` and (when used) `eps_rate`.*

## 29 Recurrence: certified vs assumed, and robustness

The gap-free upgrade separates *existence* of an algebraically contracting block (a Doeblin word on the active face) from *recurrence* of that block along trajectories. A correct finite recurrence layer is obtained by separating two objects: **(RC1)** is a purely combinatorial predicate for tight trajectories in a fixed tight SCC  $H$ , while  $\text{OG4}(H, w)$  records the explicit failure mode via an avoidance-cycle witness.

This section enforces that separation in a robust, checkable way. All recurrence statements are permitted only in one of two modes, each with an explicit robustness threshold.

### 29.1 Two recurrence modes with margins

Fix an aperiodic tight SCC  $H$  of the tight graph  $G^{(0)}$  and a certified Doeblin word  $w$  supported in  $H$ .

**Definition 29.1** (Recurrence modes for a robust certificate). *A robust certificate records recurrence information in exactly one of the following modes.*

- **Mode (combinatorial).** *The certificate records that **(RC1)** holds for  $(H, w)$  (Definition 15.1), with an explicit syndetic modulus  $L_{\text{syn}}(H, w)$  (Lemma 15.6). Equivalently, the avoidance graph  $A_{\neg w}(H)$  is acyclic.*
- **Mode (hypothesis).** *The certificate records a user-supplied recurrence hypothesis on a chosen minimizing trajectory, for example a bounded-gap hypothesis (gap bound  $B$ ) or a lower-density hypothesis (density margin  $\eta > 0$ ). In this mode, the pipeline does not certify the truth of the hypothesis; it only certifies that all constants depending on  $w$  and the template family degrade explicitly and continuously under perturbation.*

## 29.2 Robustness threshold for recurrence statements

The point of robust mode is that the same finite witness objects remain valid on an explicit neighborhood. The neighborhood size is computed from earlier margins.

**Definition 29.2** (Recurrence robustness radius). *Let  $\varepsilon_{\text{tight}}$  be the tight-core stability threshold from Section 27 and let  $\varepsilon_{\text{Doebelin}}$  be the Doeblin-survival threshold from Section 28. Define*

$$\varepsilon_{\text{rec}} := \min\{\varepsilon_{\text{tight}}, \varepsilon_{\text{Doebelin}}\}.$$

*A recurrence statement is said to be robustly certified on the neighborhood of size  $\varepsilon_{\text{rec}}$  if it is in Mode (combinatorial) and its finite witness objects are unchanged throughout that neighborhood.*

**Proposition 29.3** (Stability of the RC1/OG4 dichotomy). *Assume  $\varepsilon \leq \varepsilon_{\text{tight}}$ . Then the tight graph  $G^{(0)}$  and its SCC decomposition are unchanged under the perturbation model, so the avoidance graph  $\mathcal{A}_{\neg w}(H)$  is unchanged as a finite directed graph. Consequently, the dichotomy*

$$(RC1) \text{ holds for } (H, w) \quad \text{or} \quad OG4(H, w) \text{ holds}$$

*(and any explicit witness: either  $L_{\text{syn}}$  or an avoidance cycle) is invariant under all such perturbations.*

*Proof.* Under  $\varepsilon \leq \varepsilon_{\text{tight}}$ , the tight-core stability theorem guarantees that the tight-edge set and hence the tight SCC  $H$  are unchanged. The avoidance graph  $\mathcal{A}_{\neg w}(H)$  is constructed purely from the tight edge set of  $H$  and the fixed word  $w$  (Definition 15.2), so its vertex/edge set is unchanged. Therefore acyclicity (RC1) and the existence of an avoidance cycle (OG4) are unchanged, and any explicit witness carries over.  $\square$

**Remark 29.4** (Why  $\varepsilon_{\text{rec}}$  uses  $\varepsilon_{\text{Doebelin}}$ ). *Proposition 29.3 is purely combinatorial and only needs  $\varepsilon_{\text{tight}}$ . However, in the contraction pipeline, recurrence is meaningful only in tandem with the Doeblin minorization property of  $w$ . Thus robust certificates record  $\varepsilon_{\text{rec}} = \min\{\varepsilon_{\text{tight}}, \varepsilon_{\text{Doebelin}}\}$  so that “recurrence + Doeblin” survives together.*

## 29.3 What robust mode does and does not claim

- In Mode (combinatorial), recurrence is *derived* for *all* tight trajectories supported in the same SCC  $H$ , with an explicit gap bound  $L_{\text{syn}}$ . This derived recurrence is stable for all perturbations with  $\varepsilon \leq \varepsilon_{\text{rec}}$ .
- In Mode (hypothesis), recurrence is *not* asserted without the stated hypothesis. Robustness refers only to the *constants* in the contraction estimate (diameter bounds, Birkhoff factor, and rate degradation) when the same hypothesis is assumed along the perturbed instance.

**Enforcement rule.** No theorem, corollary, or remark may state a recurrence conclusion unless it cites *either* a Mode (combinatorial) certificate field (with  $L_{\text{syn}}$  and  $\varepsilon_{\text{rec}}$ ) *or* a Mode (hypothesis) field (with its explicit margin and the disclaimer that the hypothesis is assumed).

## 30 Statistical-stability analogue for certified extremal-growth dynamics

A recurring theme in chaos theory is *statistical stability*: observable long-run behavior should vary continuously under small perturbations of the system. In general smooth dynamics this is subtle (physical/SRB measures may bifurcate or disappear), but on the certified positive-template class studied here the pipeline produces an explicit *finite* stability mechanism.

The role of an “observable” in our setting is played by: (i) the *finite descriptor* (tight-core graph, SCC labels, admissible words, avoidance graph), and (ii) the *rate pack* controlling projective contraction on the active face. The robustness layers (robust certificate contract, tight-core stability, Doeblin stability, and the recurrence layer) are precisely the ingredients needed to state a clean stability theorem.

**Theorem 30.1** (Certified statistical stability on the certified class). *Fix a reduced descriptor instance for a finite positive-template family and suppose the pipeline produces either*

- *an obstruction witness (OG1–OG4), or*
- *a robust certificate containing (i) a tightness gap margin  $\gamma$  (Definition 26.1), (ii) a Doeblin word  $w$  with margin  $\delta > 0$ , length  $L$ , and block upper envelope  $U$  (as recorded in the robust certificate contract), and (iii) a recurrence record in one of the two modes of Definition 29.1.*

Let  $\varepsilon^* = (\varepsilon_c^*, \varepsilon_A^*)$  be the aggregate robust radius stored in the robust certificate (Definition 26.6), and assume a perturbation  $\varepsilon = (\varepsilon_c, \varepsilon_A)$  with  $\varepsilon \leq \varepsilon^*$ . Then:

- **Descriptor stability.** *The tight edge set, tight-core SCC decomposition, and every finite descriptor object constructed purely from them (in particular the admissibility of  $w$  in its SCC and the avoidance graph  $\mathcal{A}_{-w}(H)$ ) are unchanged under the perturbation. Consequently, if the unperturbed output is an obstruction witness (OG1–OG4), the same witness remains valid throughout the neighborhood.*
- **Doeblin stability with explicit degradation.** *The same word  $w$  remains a Doeblin witness on the same block with a surviving margin*

$$\delta'(\varepsilon_A) \geq \delta - L(M + \varepsilon_A)^{L-1} \varepsilon_A$$

(Lemma 28.1), and an upper envelope

$$U'(\varepsilon_A) \leq U + L(M + \varepsilon_A)^{L-1} \varepsilon_A$$

(Lemma 28.3). Hence a certified projective diameter bound and a Birkhoff contraction factor bound are available as explicit functions

$$\Delta(\varepsilon_A) \leq 2 \log \left( \frac{U'(\varepsilon_A)}{\delta'(\varepsilon_A)} \right), \quad \tau'(\varepsilon_A) \leq \tanh \left( \frac{\Delta(\varepsilon_A)}{4} \right)$$

(Lemma 28.4).

- **Recurrence stability (mode-dependent).** *In Mode (combinatorial), the RC1/OG4( $H, w$ ) dichotomy and any explicit witness (syndetic modulus or avoidance cycle) are invariant under the perturbation on the neighborhood of size  $\varepsilon_{\text{rec}}$  (Proposition 29.3). In Mode (hypothesis), no recurrence assertion is made; however, if the stated hypothesis is assumed to hold for the perturbed instance as well, then the contraction constants degrade only through the explicit functions above.*

- **Statistical-stability consequence for projective dynamics.** Whenever a recurrence modulus provides a bounded-gap parameter  $B$  for occurrences of  $w$  along the trajectory under study, one obtains for all times  $n$  the explicit forgetting estimate

$$d_H(x_n, y_n) \leq C (\tau'(\varepsilon_A))^{[n/(L+B)]} d_H(x_0, y_0)$$

(Corollary 28.6). Thus the qualitative “regularity” output (uniform projective contraction on the relevant minimizing class) is stable on the neighborhood, while the quantitative rate varies continuously with  $\varepsilon_A$  through  $\tau'(\varepsilon_A)$ .

**Remark 30.2** (How this interfaces with chaos-theory language). *Theorem 30.1 is the certified-class analogue of statistical stability: the finite descriptor and the contraction/forgetting estimates persist under small perturbations, with explicit degraded constants. In particular, on any chamber covered by a robust certificate, the long-run projective behavior of extremal trajectories is determined by finite artifacts and is stable under perturbations up to the computed radius  $\varepsilon^*$ . When the certificate fails (OG1–OG4 or margin collapse), the returned obstruction data localize the boundary where “effective description” must change.*

## 31 Sensitive vs rough dependence on initial data: a certified analogue and failure localization

One of the central “predictability” questions in chaos theory is how the future state depends on small changes of initial data. Classically, chaotic systems exhibit *sensitive dependence*: two nearby initial states separate at an exponential rate along typical trajectories. In some infinite-dimensional fluid models, an even stronger phenomenon—often called *rough dependence on initial data*—is discussed: very small perturbations can lead to large changes on very short time scales because the solution map lacks a usable uniform modulus of continuity.

In this paper the relevant dynamics is the *extremal-growth projective dynamics* on the active face, measured in the Hilbert metric. On the certified positive-template class, the witness language yields a sharp dichotomy: either one has an explicit projective forgetting mechanism (Doebelin word plus recurrence), or the pipeline returns explicit obstruction or margin-collapse data identifying where such a uniform forgetting mechanism cannot be certified.

### 31.1 A uniform projective forgetting modulus in the witnessed regime

Fix a word sequence (trajectory) and let  $x_n$  denote the normalized projective state after  $n$  steps. Whenever the recurrence layer supplies a bounded-gap parameter  $B$  for occurrences of the witnessed Doebelin word  $w$  of length  $L$  along the trajectory, the Doebelin-to-contraction machinery yields an explicit, uniform estimate of the dependence on initial data.

**Proposition 31.1** (Certified absence of sensitive dependence on the minimizing class). *Assume the pipeline outputs a robust certificate with a Doebelin word  $w$  (margin  $\delta > 0$ , length  $L$ , block upper envelope  $U$ ) and a recurrence record that provides a bounded-gap parameter  $B$  for occurrences of  $w$  along the trajectory under study. Let  $\Delta$  be the associated projective diameter bound and  $\tau = \tanh(\Delta/4)$  the corresponding Birkhoff contraction factor. Then there exists an explicit constant  $C \geq 1$  (read off from the rate pack) such that for all  $n \geq 0$  and all initial data  $x_0, y_0$  in the active face,*

$$d_H(x_n, y_n) \leq C \tau^{[n/(L+B)]} d_H(x_0, y_0).$$

In particular, the projective solution map  $x_0 \mapsto x_n$  is uniformly continuous (indeed contracting on a coarse time scale) on the minimizing class selected by the certificate.

**Remark 31.2** (Predictability horizon). *Proposition 31.1 gives an explicit predictability horizon in Hilbert metric. Given a target tolerance  $\eta > 0$ , the inequality*

$$C \tau^{\lfloor n/(L+B) \rfloor} d_H(x_0, y_0) \leq \eta$$

*can be solved for  $n$  using only the finite data  $(C, \tau, L, B)$  contained in the rate pack and recurrence record. Thus “loss of memory” is not a qualitative slogan in the witnessed regime: it is a computable bound.*

### 31.2 Failure localization: where a uniform modulus cannot persist

The witnessed forgetting estimate depends quantitatively on the Doeblin margin and on the diameter bound. In the robust layers, these appear through the functions

$$\Delta \leq 2 \log\left(\frac{U}{\delta}\right), \quad \tau \leq \tanh\left(\frac{\Delta}{4}\right),$$

and their perturbed analogues  $\Delta(\varepsilon_A)$  and  $\tau'(\varepsilon_A)$  in Theorem 30.1. Consequently, a loss of quantitative predictability can occur only near explicit margin-collapse loci.

**Proposition 31.3** (Margin collapse forces contraction loss). *Consider a sequence of certified instances for which the Doeblin margin  $\delta_k$  tends to 0 while the corresponding upper envelopes  $U_k$  remain bounded above. Then the projective diameter bounds satisfy  $\Delta_k \rightarrow \infty$  and the Birkhoff factors satisfy  $\tau_k \rightarrow 1$ . In particular, no uniform exponential forgetting rate (with a fixed  $\tau < 1$ ) can hold uniformly across such a sequence.*

**Remark 31.4** (What the pipeline returns at the frontier). *The global cover/atlas decider and the robust certificate objects are designed to identify these frontier regimes. If a uniform witnessed contraction rate cannot hold on a compact normalized family, the output is not a vague statement of complexity. Instead, the procedure returns explicit obstruction witnesses (OG1–OG4) and/or explicit degeneracy data (tightness-gap collapse, Doeblin-margin collapse, or recurrence failure certificates) that localize where and why the uniform modulus breaks down. This is the certified-class analogue of distinguishing “sensitive” and “rough” dependence: in the covered chambers one has an explicit modulus, while at the boundary the returned witnesses indicate the precise failure mechanism.*

## 32 Switching-control wedge: a two-mode example where forcing stabilizes and overuse backfires

This section gives a compact worked example showing how the certificate language translates into a familiar “control can backfire” narrative without leaving the certified class. The point is not to model an ecosystem in full fidelity, but to show how a switching policy becomes a finite symbolic object, and how the pipeline either certifies robust stabilization (via a syndetic Doeblin word) or returns a finite obstruction witness (a Doeblin-avoidance cycle).

### 32.1 Two templates and a Doeblin word

Consider two nonnegative  $2 \times 2$  templates acting on  $\mathbb{R}_{\geq 0}^2$ , interpreted as a linearized two-compartment system (for intuition: “pest” and “predator”). Fix a parameter  $\alpha \in (0, 1)$  and define

$$P := \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Template  $P$  suppresses the second component relative to the first (triangular with a strict contraction  $\alpha$  in the lower block), while  $R$  restores coupling from the first component into the second. A single-step action need not be mixing, but the two-step word

$$w := RP$$

produces a strictly positive block:

$$RP = \begin{pmatrix} 1 & 1 \\ 1 & 1 + \alpha \end{pmatrix},$$

so  $w$  is a valid Doeblin word on the full face. In the certificate language, the Doeblin search can legitimately return  $w$  as a witnessed mixing word for the active face.

### 32.2 The avoidance-cycle obstruction (why overuse can defeat mixing)

The existence of a Doeblin word does *not* imply it recurs along every tight trajectory. Indeed, the switching policy “apply  $P$  forever” generates products  $P^n$  which remain upper-triangular and never contain  $w = RP$ . In the recurrence layer, this is exactly the obstruction  $\text{OG4}(H, w)$ .

To see the finite witness concretely, consider the tight SCC  $H$  on a two-vertex abstraction  $\{h, \ell\}$  (“high predator” and “low predator” states) with edges

$$\ell \xrightarrow{R} h, \quad h \xrightarrow{P} \ell, \quad \ell \xrightarrow{P} \ell.$$

The word  $w$  is the length-2 cycle  $\ell \xrightarrow{R} h \xrightarrow{P} \ell$ . However, the self-loop  $\ell \xrightarrow{P} \ell$  allows an infinite tight trajectory that avoids  $w$  forever. In the avoidance graph  $\mathbf{A}(H, w)$  (Definition 15.2) for  $m = 2$ , this appears as a directed cycle at the vertex  $(0, \ell)$ :

$$(0, \ell) \rightarrow (0, \ell) \quad \text{via the edge } \ell \xrightarrow{P} \ell.$$

By Proposition 18.2, this cycle is an explicit minimizing trajectory with no Doeblin blocks, so no statement of uniform projective contraction can hold for *all* tight minimizers in this SCC. Operationally, the pipeline would return  $\text{OG4}$  with this one-edge witness.

This is the certified analogue of “pesticide paradox” timing: a control action that is locally beneficial (a single  $P$  step) can, when overused, trap the dynamics in a regime where the global stabilizing pattern  $w = RP$  never occurs.

### 32.3 A stabilizing forcing rule and a certified rate pack

The same finite witness also tells you how to force stability. If the control contract forbids the overuse loop  $\ell \xrightarrow{P} \ell$  (do not apply  $P$  when already at  $\ell$ ), then the SCC becomes the two-cycle

$$\ell \xrightarrow{R} h \xrightarrow{P} \ell,$$

and every infinite trajectory contains  $w = RP$  with gap exactly 2. Equivalently, the avoidance graph becomes acyclic, so  $(H, w)$  satisfies (RC1) (Definition 15.1) with

$$L_{\text{syn}}(H, w) = 2.$$

In this constrained regime the Doeblin word returned by the pipeline is syndetic by construction, and Theorem 19.7 together with the rate-pack extraction in §20 yields an explicit Hilbert-metric contraction rate. Moreover, by the robustness layer (§28 and §30), this stabilizing conclusion persists under small perturbations of the template entries within the computed radius  $\varepsilon^*$ .

**Takeaway.** The certificate dichotomy does not merely say “mixing exists”; it outputs a finite obstruction when mixing can be avoided, and when it cannot, it outputs a finite recurrence modulus and an explicit quantitative contraction guarantee. This is precisely the kind of effective, checkable “control vs backfire” resolution that open-problem lists in chaos theory ask for, but carried out on a regime where all claims are decidable and robust.

### 33 Approximate Certification on Parameter Regions (Robust Neighborhood Cover)

This section upgrades pointwise certificates to *region certification*. The guiding observation is topological and quantitative: all certificate predicates used by the pipeline are *open* conditions once the robust margins are recorded. Therefore a finite verification on an  $\varepsilon$ -net certifies an entire parameter region except near an explicit exceptional set where margins vanish.

#### 33.1 Local openness packaged as a single statement

Fix a parameter instance  $\theta$  (costs and templates) and assume the pipeline produces a robust certificate object with robustness radii  $\varepsilon_{\text{tight}}, \varepsilon_{\text{Doeblin}}, \varepsilon_{\text{rec}}$  and  $\varepsilon_* = \min\{\varepsilon_{\text{tight}}, \varepsilon_{\text{Doeblin}}, \varepsilon_{\text{rate}}, \varepsilon_{\text{rec}}\}$  as in Sections 26–29.

**Theorem 33.1** (Robust neighborhood invariance). *Let  $\theta$  admit a robust certificate with radius  $\varepsilon_* > 0$ . Then every perturbed instance  $\theta'$  with perturbation size  $\leq \varepsilon_*$  satisfies:*

- *the tight graph  $G^{(0)}$  and its SCC/period decomposition are unchanged;*
- *the stored Doeblin word  $w$  remains a valid Doeblin witness on the same active face, with explicit margin  $\delta'(\varepsilon_*) > 0$ ;*
- *the recurrence layer is unchanged in Mode (combinatorial) (same RC1/OG4 outcome and witnesses), or remains a stated hypothesis in Mode (hypothesis);*
- *the projective contraction constants degrade explicitly according to the formulas stored in the certificate.*

*In particular, every theorem in this paper that is invoked through certificate fields at  $\theta$  holds verbatim at  $\theta'$  with constants replaced by the certificate’s degraded constants.*

*Proof.* Tight-core invariance is Theorem 27.2. Doeblin survival and the explicit  $\delta'(\varepsilon) / \tau'(\varepsilon)$  bounds are established in Section 28. Recurrence-layer stability in Mode (combinatorial) is the RC1/OG4 equivalence and its robustness statement in Section 29. All remaining constants are explicit functions of these margins and the FTG bounds, hence are stable under the same inequalities.  $\square$

### 33.2 Finite net certification of a compact region

Let  $(\Omega, d)$  be a compact parameter set with a product metric compatible with the perturbation model (e.g.  $d(\theta, \theta') = \max\{\|\Delta c\|_\infty, \|\Delta A\|_{\infty, \text{entry}}\}$  in absolute-entrywise mode).

**Definition 33.2** ( $\varepsilon$ -net). *A finite set  $\mathcal{N} \subset \Omega$  is an  $\varepsilon$ -net if every  $\theta \in \Omega$  lies within distance  $\varepsilon$  of some  $\theta_i \in \mathcal{N}$ .*

**Theorem 33.3** (Region certification by robust balls). *Suppose that for each net point  $\theta_i \in \mathcal{N}$  the pipeline produces a robust certificate with radius  $\varepsilon_*(\theta_i) > 0$ . Define the certified region*

$$\Omega_{\text{cert}} := \bigcup_{\theta_i \in \mathcal{N}} B(\theta_i, \varepsilon_*(\theta_i)) \cap \Omega.$$

*Then every instance  $\theta \in \Omega_{\text{cert}}$  inherits the same certified conclusions as its center  $\theta_i$  (in the sense of Theorem 33.1). Moreover, if  $\mathcal{N}$  is an  $\varepsilon$ -net and  $\min_i \varepsilon_*(\theta_i) \geq \varepsilon$ , then  $\Omega_{\text{cert}} = \Omega$ .*

*Proof.* If  $\theta \in \Omega_{\text{cert}}$  then  $\theta \in B(\theta_i, \varepsilon_*(\theta_i))$  for some center  $\theta_i$ . Apply Theorem 33.1 at  $\theta_i$ . The final claim follows directly from the definition of an  $\varepsilon$ -net.  $\square$

### 33.3 Exceptional set description

The only ways robustness can fail are the explicit margin-boundary events where  $\varepsilon_* = 0$ . Accordingly, region mode reports uncovered points as being near one of the following certificate-boundary loci:

- **Tightness degeneracy:**  $\gamma = 0$  (tight/non-tight separation vanishes), so the tight graph can change.
- **Doebelin degeneracy:**  $\delta = 0$  for the stored word (or no Doebelin word exists), so minorization can fail.
- **Recurrence boundary:** the RC1/OG4 dichotomy changes only when the tight graph changes (hence is subsumed by  $\gamma = 0$ ), while Mode (hypothesis) has no derived recurrence boundary because recurrence is not asserted.

### 33.4 Pipeline: region mode outputs

In region mode, the pipeline:

- samples or constructs a finite net  $\mathcal{N}$  of parameter points;
- runs the pointwise certificate pipeline at each  $\theta_i$ ;
- outputs the list of certified balls  $(\theta_i, \varepsilon_*(\theta_i))$  and their conclusions;
- reports uncovered points and diagnoses them in terms of the margin boundaries above.

A recommended machine-readable artifact is a `robust_region_report.json` containing: centers with their radii  $\varepsilon_*$ , the induced outcome labels (OG1–OG4 or contraction), and an uncovered-set diagnostic summary.

## 34 From regional robustness to cover/atlas universality on compact normalized families

The robustness layer establishes a local principle: whenever a robust certificate exists at a parameter value  $\theta$ , the same conclusions hold on an explicit neighborhood around  $\theta$ , with conservatively degraded constants. This section records what is required to upgrade from “open neighborhoods away from degeneracy” to a global statement on a compact normalized parameter family.

**Terminology.** Throughout this section, “global” means *on a fixed compact normalized family*  $\Theta$ , and “universality” is always understood *within the certified class* in the *cover-or-atlas* sense: either good-global contraction is certified, or an explicit boundary-atlas report localizes the obstruction/degeneracy set.

### 34.1 Two global targets

There are two distinct global goals.

- **Global regime decidability (already achieved on the certified class).** For any instance in the certified class, the pipeline halts and returns a finite outcome object: *either* a robust contraction certificate, *or* a concrete obstruction witness (OG1–OG4).
- **Good-global contraction universality (stronger).** On a parameter region  $\Theta$ , every parameter point lies in the contraction regime, so the exceptional set is empty and one can state a single uniform robustness radius on all of  $\Theta$ .

The remainder of this section concerns the second target.

### 34.2 Paired radii convention

In the robust certificate contract, perturbations may act on two layers: costs and template entries. Accordingly, the local robustness radius naturally comes as a pair

$$\epsilon^*(\theta) := (\epsilon_c^*(\theta), \epsilon_A^*(\theta)),$$

meaning that the certificate remains valid whenever both perturbations satisfy  $\|\Delta c\| \leq \epsilon_c^*(\theta)$  and  $\|\Delta A\| \leq \epsilon_A^*(\theta)$  in the perturbation metric fixed by the robust certificate contract. For compactness and cover arguments, it is often convenient to also record the conservative scalar floor

$$\epsilon_{\min}^*(\theta) := \min\{\epsilon_c^*(\theta), \epsilon_A^*(\theta)\}.$$

This scalar floor is sufficient for a single-radius cover statement, while the pair is the natural artifact for downstream reproducibility.

### 34.3 Uniform margins and a global contraction theorem

Let  $\Theta$  be a compact parameter region. For each  $\theta \in \Theta$ , the reduction stage produces a tight graph  $H(\theta)$ , face data, and (when obstruction gates do not fire) a Doeblin witness word  $w(\theta)$  on an aperiodic tight SCC.

**Definition 34.1** (Uniform robust margins on a region). *We say that  $\Theta$  admits uniform robust margins for a fixed witness word  $w$  if there exist constants  $\gamma_0, \delta_0 > 0$  and a fixed aperiodic tight SCC  $H$  such that for every  $\theta \in \Theta$ :*

- the tightness gap satisfies  $\gamma(\theta) \geq \gamma_0$  and the tight-core graph equals  $H(\theta) = H$ ;
- the Doeblin witness product along  $w$  satisfies  $\min(P_w(\theta)) \geq \delta_0$  on the active coordinates;
- the recurrence forcing predicate *RC1* holds on  $H$  for the word  $w$  (equivalently: the tight-language avoidance automaton contains no directed cycle).

**Theorem 34.2** (Good-global contraction universality on  $\Theta$  from uniform margins). *Assume  $\Theta$  is compact and normalized. If  $\Theta$  admits uniform robust margins in the sense of Definition 34.1 for a fixed Doeblin witness word  $w$ , then there exist:*

- a paired uniform robustness radius  $\varepsilon_{\text{global}} = (\varepsilon_{c,\text{global}}, \varepsilon_{A,\text{global}})$ ,
- a conservative scalar floor  $\varepsilon_{\min,\text{global}} := \min\{\varepsilon_{c,\text{global}}, \varepsilon_{A,\text{global}}\} > 0$ ,
- and a uniform rate pack  $(\tau_{\text{global}}, C_{\text{global}}, \kappa_{\text{global}})$ ,

with the following property. For every  $\theta \in \Theta$ :

- every tight trajectory in the aperiodic SCC  $H$  experiences Doeblin blocks with a bounded-gap modulus that is uniform over  $\Theta$ ;
- consequently, the associated projective dynamics contract uniformly with rate  $\tau_{\text{global}} < 1$  and prefactor  $C_{\text{global}}$ ;
- the same conclusions remain valid under all perturbations satisfying  $\|\Delta c\| \leq \varepsilon_{c,\text{global}}$  and  $\|\Delta A\| \leq \varepsilon_{A,\text{global}}$  (in the contract's perturbation metric), and in particular for all perturbations of size at most  $\varepsilon_{\min,\text{global}}$  in either layer.

Moreover,  $\varepsilon_{\text{global}}$  is computable from the uniform margins  $(\gamma_0, \delta_0)$  and the fixed witness metadata (word length and uniform entry bounds) by taking the componentwise minimum of the explicit thresholds from the tight-core stability and Doeblin stability layers.

*Proof.* By the tight-core stability theorem, any perturbation smaller than the explicit threshold determined by  $\gamma_0$  preserves the tightness classification and the SCC decomposition. By Doeblin stability, the fixed Doeblin product along  $w$  with entry floor  $\delta_0$  persists under perturbations below the explicit telescoping-product threshold, yielding a degraded floor  $\delta'_0$ . Since  $H$  and  $w$  are fixed throughout  $\Theta$ , the avoidance automaton used to encode RC1 is fixed. RC1 holding implies that this finite automaton has no directed cycle; hence every tight trajectory must hit the Doeblin block within at most  $L_{\text{syn}}(H, w) \leq |V(A(H, w))|$  steps (Corollary 15.7), giving a uniform recurrence modulus. Combining the recurrence modulus with the Hilbert/Birkhoff contraction lemma and the explicit rate pack yields uniform projective contraction with constants depending only on  $\delta'_0$  and the certificate geometry. Taking the componentwise minimum of the explicit tight-core and Doeblin thresholds produces a paired radius valid for all  $\theta \in \Theta$ .  $\square$

### 34.4 A global theorem phrased in terms of degeneracy avoidance

The uniform-margin hypothesis is clean, but one still wants a practical route to verify it. A useful middle step is to express good-global contraction universality on compact normalized  $\Theta$  within the certified class as the absence of certain degeneracy surfaces.

**Definition 34.3** (Degeneracy sets). *Fix a witness word  $w$  and consider a region  $\Theta$  on which the tight SCC  $H$  and the active coordinates are well-defined. Define the following subsets of  $\Theta$ :*

- $\mathcal{D}_\gamma := \{\theta \in \Theta : \gamma(\theta) = 0\}$  (*tightness degeneracy / tie surfaces*),
- $\mathcal{D}_\delta(w) := \{\theta \in \Theta : \min(P_w(\theta)) = 0\}$  (*loss of strict positivity for the witness product*),
- $\mathcal{D}_{\text{RC1}}(w) := \{\theta \in \Theta : \text{RC1 fails for } (H, w)\}$  (*recurrence-forcing failure*).

**Proposition 34.4** (Good-global contraction universality from empty degeneracy on  $\Theta$ ). *Assume  $\Theta$  is compact and that  $H$  and  $w$  are constant on  $\Theta$ . If*

$$\mathcal{D}_\gamma = \emptyset, \quad \mathcal{D}_\delta(w) = \emptyset, \quad \mathcal{D}_{\text{RC1}}(w) = \emptyset,$$

*then  $\Theta$  admits uniform robust margins and Theorem 34.2 applies. In particular,  $\gamma$  and  $\theta \mapsto \min(P_w(\theta))$  have strictly positive minima on  $\Theta$ .*

*Proof.* On a region where  $H$  and  $w$  are fixed,  $\gamma(\theta)$  is the minimum of finitely many strict reduced-cost separations among edges in the fixed reduced instance, hence is continuous. The map  $\theta \mapsto \min(P_w(\theta))$  is continuous because  $P_w(\theta)$  is a finite product of matrices whose entries depend continuously on  $\theta$ . On a compact set, each continuous function attains its minimum. If the corresponding zero set is empty, the minimum must be strictly positive, giving constants  $\gamma_0, \delta_0 > 0$ . RC1 being satisfied everywhere on  $\Theta$  provides the recurrence-forcing item in Definition 34.1.  $\square$

For the linear parameter families treated in §38, the parameter space decomposes into finitely many polyhedral chambers on which the tight graph and the witness metadata are constant. In that setting, Proposition 34.4 can be applied chamberwise, and good-global contraction universality on compact normalized  $\Theta$  within the certified class reduces to taking minima over the finitely many certified chambers.

### 34.5 Frontier atlas: chamber inequalities and boundary loci

The cover/atlas output is most useful when the “frontier” is not a slogan but a concrete, checkable set of inequalities. In the certified positive-template class, the relevant frontier is the union of two kinds of boundaries: (i) *combinatorial boundaries* where the tight geometry changes, and (ii) *robustness boundaries* where a fixed witness loses its strict margins.

**Chambers in linear families (explicit inequalities).** In the linear cycle-mean parameter model of §38, the basic chambers are the connected components of the complement of the degeneracy arrangement  $\mathcal{D}$ . Equivalently, a chamber is specified by a constant sign pattern of the finitely many affine functionals  $F_{\gamma, \gamma'}(\theta) = \mu_\theta(\gamma) - \mu_\theta(\gamma')$ , hence by a finite system of strict linear inequalities  $F_{\gamma, \gamma'}(\theta) < 0$  and  $F_{\gamma, \gamma'}(\theta) > 0$ . On each such chamber the minimizing cycle set (and therefore the tight SCC data) are constant.

**Robust witness chambers (add the margin inequalities).** Once the witness data are frozen (Definition 34.17), there are two additional strict inequalities that separate certified chambers from boundary pathology:

- **Tightness gap:**  $\gamma(\theta) > 0$ , equivalently  $\theta \notin \mathcal{D}_\gamma$ .
- **Doebelin positivity gap:**  $\delta_w(\theta) = \min(P_w(\theta)) > 0$ , equivalently  $\theta \notin \mathcal{D}_\delta(w)$ .

Since RC1 is a purely finite automaton predicate once  $(H, w)$  are fixed (Lemma 34.15), recurrence forcing does not introduce an additional analytic boundary inside such a chamber.

**The frontier as a finite union of named loci.** Within a frozen witness chamber, the full “bad set” that prevents good-global contraction universality is contained in the named union

$$\mathcal{F}(H, w) := (\mathcal{D} \cap \Theta) \cup \mathcal{D}_\gamma \cup \mathcal{D}_\delta(w) \cup \mathcal{D}_{\text{RC1}}(w),$$

where  $\mathcal{D}$  is the hyperplane arrangement from §38 (in linear families) and the remaining sets are as in Definition 34.3. The practical meaning is: away from  $\mathcal{F}(H, w)$ , the entire certificate stack is stable and the rate pack varies only by the explicit degradation formulas.

**Definition 34.5** (Frontier atlas on a compact normalized family). *Fix a compact normalized parameter family  $\Theta$  within the certified class. A frontier atlas for  $\Theta$  is a finite list of labeled pieces  $\{\Omega_j\}_{j=1}^J$  with the following data attached to each piece:*

- a witness package (frozen tight SCC  $H_j$ , witness word  $w_j$ , and mode labels),
- a coverage certificate (either a robust ball  $B(\theta_i, \varepsilon^*(\theta_i))$  as in the region certification theorem, or a chamber description in the linear-family model),
- either (**good**) certified lower bounds  $\gamma_0(\Omega_j) > 0$ ,  $\delta_0(\Omega_j) > 0$  and an RC1 confirmation (hence a chamberwise rate pack),
- or (**boundary**) a near-degeneracy label specifying which named locus in  $\mathcal{F}(H_j, w_j)$  is responsible (e.g. “ $\gamma \rightarrow 0$ ” or “ $\delta \rightarrow 0$ ”), together with the corresponding finite obstruction/diagnostic payload.

The atlas is certified if the union of the coverage certificates contains  $\Theta$ .

**Checklist for a chamberwise “good” label.** In practice, Definition 34.5 amounts to a short finite checklist on each piece:

- verify frozen witness data  $(H, w)$  (tight-core invariance),
- verify  $\gamma_0 > 0$  and  $\delta_0 > 0$  by a net + Lipschitz envelope (Proposition 34.14) or by direct minima on a finite chamber description,
- verify RC1 by cycle detection in the avoidance automaton (Lemma 34.15),
- emit the degraded-constant formulas and resulting rate pack.

When the checklist cannot be completed, the atlas records the first failing item as a named boundary label and returns the corresponding finite witness/diagnostic object.

### 34.6 Region-mode coverage as a certificate of good-global contraction on compact normalized families

The region certification theorem provides a covering principle: each robust certificate at  $\theta$  certifies a whole neighborhood around  $\theta$ . A global statement on  $\Theta$  follows once the union of certified neighborhoods covers  $\Theta$ .

**Proposition 34.6** (Good-global contraction certificate via region coverage). *Assume  $\Theta$  is compact and normalized. Suppose region mode produces a finite family of robust certificates  $\{\mathcal{C}_i\}$  with certified paired radii  $\varepsilon^*(\mathcal{C}_i)$  such that*

$$\Theta \subset \bigcup_i B(\theta_i, \varepsilon_{\min}^*(\mathcal{C}_i)),$$

where  $B(\theta, \varepsilon)$  denotes the ball in parameter space used for sampling and  $\varepsilon_{\min}^*$  is the conservative scalar floor. Then  $\Theta$  admits a good-global contraction certificate within the certified class on the compact normalized family  $\Theta$ . One may set

$$\gamma_0 := \min_i \gamma(\mathcal{C}_i), \quad \delta_0 := \min_i \delta(\mathcal{C}_i), \quad \varepsilon_{\text{global}} := (\min_i \varepsilon_c^*(\mathcal{C}_i), \min_i \varepsilon_A^*(\mathcal{C}_i)),$$

and conclude uniform contraction on all of  $\Theta$  with explicit degraded constants.

*Proof.* The conclusion is exactly the region certification theorem applied to the finite subcover exhibited by region mode. Uniformity follows by taking minima over the finite family.  $\square$

**Proposition 34.7** (Finite termination of region mode under a uniform scalar floor). *Assume  $\Theta \subset \mathbb{R}^d$  is compact and normalized and admits a uniform lower bound*

$$\inf_{\theta \in \Theta} \varepsilon_{\min}^*(\theta) \geq \varepsilon_0 > 0$$

*for the conservative scalar floors returned by the pipeline. Then a grid net of mesh  $\varepsilon_0/2$  produces a finite cover, hence region mode terminates with a good-global contraction certificate within the certified class on the compact normalized family  $\Theta$ .*

*Proof.* Compactness implies  $\Theta$  can be covered by finitely many balls of radius  $\varepsilon_0/2$ . At each ball center, the pipeline returns a certificate with scalar floor at least  $\varepsilon_0$ . Therefore each such certificate neighborhood contains the corresponding grid ball, giving a finite cover of  $\Theta$ .  $\square$

### 34.7 A practical push: hybrid chamberwise verification

The most practical route to good-global contraction universality on compact normalized families (within the certified class) is a hybrid procedure:

- use the chamber mechanism to identify regions on which  $(H, w)$  are constant,
- on each chamber, either prove degeneracy sets are empty (Proposition 34.4), or close coverage via region mode (Proposition 34.6),
- if coverage fails, the uncovered pieces are automatically labeled as “near degeneracy” and come with obstruction witnesses.

The key point is that the framework never hides the boundary: good-global contraction universality on compact normalized  $\Theta$  within the certified class is exactly the claim that all named degeneracies are absent on  $\Theta$ . When they are present, the pipeline returns a concrete finite witness locating them.

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**Algorithm 9** Hybrid cover/atlas check on compact normalized  $\Theta$  within the certified class (certificate-driven)

---

**Require:** Compact region  $\Theta$  (parameterization), perturbation metric (robust certificate contract)

**Ensure:** Either a GlobalUniversalityReport for compact normalized  $\Theta$  within the certified class, or an explicit near-degeneracy/obstruction report

- 1: Partition  $\Theta$  into chambers where  $(H, w)$  are constant (by the tight-core/chamber mechanism)
  - 2: **for** each chamber  $\Omega$  **do**
  - 3:     Run region mode on  $\Omega$  with adaptive refinement
  - 4:     **if** coverage closes **then**
  - 5:         Record chamberwise global bounds  $\gamma_0(\Omega), \delta_0(\Omega), \varepsilon_{\text{global}}(\Omega)$
  - 6:     **else**
  - 7:         Record uncovered set and its near-degeneracy label (tie surface, positivity collapse, or RC1 failure)
  - 8:     **end if**
  - 9: **end for**
  - 10: If all chambers close coverage, take minima of chamberwise bounds to form a global report on  $\Theta$
- 

**Remark 34.8** (What is and is not proved “globally”). *The results above provide two globally valid statements.*

- *On the certified class, the pipeline provides a global decidability frontier: it always returns a finite obstruction or a finite robust contraction certificate.*
- *On a parameter family  $\Theta$ , a good-global contraction universality statement is achieved when either uniform margins are proved or region-mode coverage closes.*

*If neither occurs, the output is still constructive: one gets an explicit description of where and how universality can fail.*

### 34.8 Beyond compact families: normalization for cover/atlas universality within the certified class

When one asks for cover/atlas universality beyond a fixed compact normalized family, the obstruction is usually not conceptual but topological: the raw parameter space is typically *noncompact* (because of scaling degrees of freedom, unbounded costs, or templates whose entries can blow up or collapse to zero). The cover/atlas conclusions above are therefore best viewed as *compact* statements within the certified class.

In many positive-template models, however, the noncompactness is largely artificial. Two standard reductions convert noncompact questions into compact ones:

- **Projectivization/scaling.** The extremal growth rate is homogeneous under simultaneous scaling of templates. Working in projective coordinates (or fixing a canonical scale) removes this degree of freedom.
- **Canonical normalization.** One may fix a normalization such as  $\max_{i,j} A_k(i, j) = 1$  (entry-wise), or  $\|A_k\|_1 = 1$  (column-stochastic scaling), provided the admissible class is closed under the chosen normalization and the pipeline invariants are scale-invariant.

The practical upshot is that a cover/atlas universality theorem can often be stated as a compact cover/atlas universality theorem on a normalized parameter space within the certified class, together with an explicit description of the boundary faces where normalization breaks (typically exactly the named degeneracies  $\gamma = 0$  or  $\delta = 0$ ).

**Proposition 34.9** (Cover/atlas universality via compact normalization within the certified class). *Suppose the admissible parameter space  $\Theta_{\text{raw}}$  admits a continuous normalization map*

$$\mathcal{N} : \Theta_{\text{raw}} \rightarrow \Theta_{\text{norm}}$$

*into a compact set  $\Theta_{\text{norm}}$  such that:*

- *the pipeline predicates (tightness, Doeblin positivity, and the RC1 automaton) are invariant under the normalization, and*
- *the robust margins  $(\gamma, \delta)$  and the paired robust radius  $\epsilon^*$  can be chosen as functions of  $\mathcal{N}(\theta)$ .*

*If  $\Theta_{\text{norm}}$  admits a good-global contraction certificate within the certified class on the compact normalized family  $\Theta_{\text{norm}}$  (for example by Proposition 34.4 or by closing region-mode coverage as in Proposition 34.6), then the corresponding universality conclusion holds for all parameters  $\theta \in \Theta_{\text{raw}}$  whose normalization is defined. Moreover, the complement is explicitly contained in the normalization boundary, which is a subset of the named degeneracy loci.*

*Proof.* By hypothesis, the decision predicates and the robustness radii depend only on the normalized parameters. Therefore the uniform contraction conclusion on  $\Theta_{\text{norm}}$  pulls back to  $\Theta_{\text{raw}}$  wherever  $\mathcal{N}$  is defined. The only possible failure modes are points where normalization is undefined or discontinuous, which in positive-template settings correspond to entries collapsing to zero or to tightness degeneracies; these are contained in the exceptional set described earlier.  $\square$

**Remark 34.10** (What remains to claim a truly “global” theorem). *Proposition 34.9 turns the remaining work into two concrete tasks.*

- *Verify that your chosen normalization  $\mathcal{N}$  preserves the admissible class and the pipeline invariants.*
- *Prove (or certify) good-global contraction universality within the certified class on the compact normalized space  $\Theta_{\text{norm}}$ .*

*The second task is exactly the compact problem solved by the chamberwise + region-mode machinery. The first task is model-dependent but typically elementary (it is an algebraic check once the normalization is fixed).*

### 34.9 Validated global margins from finite sampling

Theorem 34.2 becomes genuinely global once one has uniform lower bounds  $\gamma_0, \delta_0$  on the margins across  $\Theta$ . When  $\Theta$  is a parameter family, a standard way to obtain such bounds is to combine a finite net with a conservative Lipschitz envelope. This subsection records a clean “verification lemma” that can be suitable for citation and implemented by a third party.

**Definition 34.11** (Net mesh). *Let  $\Theta \subset \mathbb{R}^d$  be compact and let  $\|\cdot\|$  be a fixed norm. A finite set  $\mathcal{N} \subset \Theta$  is an  $h$ -net if for every  $\theta \in \Theta$  there exists  $\theta_i \in \mathcal{N}$  with  $\|\theta - \theta_i\| \leq h$ .*

**Lemma 34.12** (A conservative net-size bound in  $\|\cdot\|_\infty$ ). *Let  $\Theta \subset \mathbb{R}^d$  be compact and define its coordinate spans  $S_j := \sup_{\theta, \theta' \in \Theta} |\theta_j - \theta'_j|$  and  $D_\infty := \max_{1 \leq j \leq d} S_j$ . For any  $h > 0$  there exists an  $h$ -net  $\mathcal{N} \subset \Theta$  (in  $\|\cdot\|_\infty$ ) with*

$$|\mathcal{N}| \leq \prod_{j=1}^d \left( \left\lceil \frac{S_j}{h} \right\rceil + 1 \right) \leq \left( \left\lceil \frac{D_\infty}{h} \right\rceil + 1 \right)^d.$$

*Proof.* Let  $B := \prod_{j=1}^d [a_j, b_j]$  be an axis-aligned box containing  $\Theta$ , with  $b_j - a_j = S_j$ . Partition each interval  $[a_j, b_j]$  into subintervals of length at most  $h$ , producing at most  $\lceil S_j/h \rceil + 1$  grid points in direction  $j$ . The resulting Cartesian grid has at most  $\prod_j (\lceil S_j/h \rceil + 1)$  points, and every point in  $B$  lies within  $\|\cdot\|_\infty$ -distance  $h$  of some grid point. Selecting, for each grid cell that intersects  $\Theta$ , a representative point of  $\Theta$  in that cell yields an  $h$ -net contained in  $\Theta$  with the same cardinality bound. The final inequality follows from  $S_j \leq D_\infty$ .  $\square$

**Lemma 34.13** (Certified minima from a Lipschitz envelope). *Let  $f : \Theta \rightarrow \mathbb{R}$  be Lipschitz with constant  $L_f$  with respect to  $\|\cdot\|$ . Let  $\mathcal{N}$  be an  $h$ -net in  $\Theta$ . Then*

$$\min_{\theta \in \Theta} f(\theta) \geq \min_{\theta_i \in \mathcal{N}} f(\theta_i) - L_f h.$$

*In particular, if the right-hand side is strictly positive then  $f(\theta) > 0$  for all  $\theta \in \Theta$ .*

*Proof.* Fix  $\theta \in \Theta$  and pick  $\theta_i \in \mathcal{N}$  with  $\|\theta - \theta_i\| \leq h$ . Lipschitz continuity gives  $f(\theta) \geq f(\theta_i) - L_f h$ . Taking the minimum over  $\theta \in \Theta$  and then the minimum over  $\theta_i \in \mathcal{N}$  yields the claim.  $\square$

**Proposition 34.14** (Uniform margin verification by net + Lipschitz bounds). *Assume  $\Theta$  is compact. Fix a candidate tight SCC  $H$  and a candidate witness word  $w$  and assume these are constant on  $\Theta$ . Assume:*

- the tightness gap function  $\gamma(\theta)$  is Lipschitz with constant  $L_\gamma$ ,
- the Doeblin margin function  $\delta_w(\theta) := \min(P_w(\theta))$  is Lipschitz with constant  $L_\delta$ ,
- RC1 holds for  $(H, w)$  (equivalently: the avoidance automaton has no directed cycle).

*Let  $\mathcal{N}$  be an  $h$ -net in  $\Theta$ . Define certified lower bounds*

$$\underline{\gamma}_0 := \min_{\theta_i \in \mathcal{N}} \gamma(\theta_i) - L_\gamma h, \quad \underline{\delta}_0 := \min_{\theta_i \in \mathcal{N}} \delta_w(\theta_i) - L_\delta h.$$

*If  $\underline{\gamma}_0 > 0$  and  $\underline{\delta}_0 > 0$ , then  $\Theta$  admits uniform robust margins in the sense of Definition 34.1, hence Theorem 34.2 applies with  $\gamma_0 = \underline{\gamma}_0$  and  $\delta_0 = \underline{\delta}_0$ .*

**Lemma 34.15** (RC1 is chamberwise constant). *Fix a candidate tight SCC  $H$  and a candidate witness word  $w$ . Let  $\mathcal{A}(H, w)$  denote the tight-language avoidance automaton used to encode RC1. If  $H$  and  $w$  are fixed on  $\Theta$ , then  $\mathcal{A}(H, w)$  is fixed on  $\Theta$ . Consequently, either RC1 holds for all  $\theta \in \Theta$  or RC1 fails for all  $\theta \in \Theta$ . In particular, RC1 can be verified at a single parameter value once the chamber is fixed.*

*Proof.* RC1 is equivalent to the absence of a directed cycle in  $\mathcal{A}(H, w)$ . When  $H$  and  $w$  are fixed, the transition structure of  $\mathcal{A}(H, w)$  does not depend on  $\theta$ . Thus the cycle predicate is independent of  $\theta$ , proving the claim.  $\square$

*Proof of Proposition 34.14.* Lemma 34.13 applied to  $f = \gamma$  and  $f = \delta_w$  yields  $\gamma(\theta) \geq \underline{\gamma}_0$  and  $\delta_w(\theta) \geq \underline{\delta}_0$  for all  $\theta \in \Theta$ . RC1 supplies the recurrence-forcing item. Thus Definition 34.1 holds with the certified bounds.  $\square$

**Remark 34.16** (How to obtain conservative Lipschitz constants). *The proposition treats  $L_\gamma$  and  $L_\delta$  as inputs. In applications, one bounds these constants from the parameterization. For example, if each template map  $\theta \mapsto A_k(\theta)$  is Lipschitz in operator  $\|\cdot\|_\infty$  with constant  $L_A$  and  $\max_{\theta \in \Theta} \|A_k(\theta)\|_\infty \leq M$ , then for a fixed word  $w$  of length  $L$  one has the conservative envelope*

$$\|P_w(\theta) - P_w(\theta')\|_\infty \leq L L_A M^{L-1} \|\theta - \theta'\|,$$

*so one may take  $L_\delta \leq L L_A M^{L-1}$ . For  $\gamma$ , a corresponding bound can be obtained once the reduced-cost computation is expressed as a finite composition of linear maps and maxima/minima whose coefficients are bounded on  $\Theta$ ; the pipeline may record a conservative  $L_\gamma$  as part of the calibration contract. At the artifact level, the parameter-family descriptor may declare the needed moduli explicitly via the `net_certification_contract` block in `theta_family_schema.json`.*

### 34.10 Explicit Lipschitz constants for the margins on a fixed witness chamber

On a chamber where the witness data are frozen, one can make the verification inputs  $L_\gamma$  and  $L_\delta$  completely explicit. The purpose of this subsection is to record conservative formulas that can be cited without invoking any hidden regularity assumptions.

**Definition 34.17** (Chamberwise frozen witness package). *Let  $\Theta \subset \mathbb{R}^d$  be a compact parameter region. We say that  $\Theta$  lies in a frozen witness chamber if there exist:*

- *a fixed calibrated pair  $(u, \lambda_*)$ ,*
- *a fixed tight edge set  $H \subseteq E$  defined by the sign test for  $c_u$  using  $(u, \lambda_*)$ ,*
- *a fixed Doeblin witness word  $w$  of length  $L$ ,*

*such that all certificates on  $\Theta$  are evaluated using this same data. In particular, the margin functions are  $\gamma(\theta) = \min\{c_u(\theta, e) : e \in E \setminus H\}$  and  $\delta_w(\theta) = \min(P_w(\theta))$  with  $H$  and  $w$  fixed.*

**Lemma 34.18** (Lipschitz constant for the tightness gap under cost parameterization). *Assume  $\Theta$  lies in a frozen witness chamber in the sense of Definition 34.17. Assume the cost map  $\theta \mapsto c(\theta) \in \mathbb{R}^E$  is Lipschitz in  $\|\cdot\|_\infty$  with constant  $L_c$ , i.e.  $\|c(\theta) - c(\theta')\|_\infty \leq L_c \|\theta - \theta'\|$  for all  $\theta, \theta' \in \Theta$ . Keep  $(u, \lambda_*)$  fixed. Then every reduced cost  $c_u(\theta, e)$  is Lipschitz with constant  $L_c$ , and the tightness gap  $\gamma(\theta) = \min_{e \in E \setminus H} c_u(\theta, e)$  is Lipschitz with constant*

$$L_\gamma \leq L_c.$$

*Proof.* For fixed  $(u, \lambda_*)$ , the map  $c \mapsto c_u(e) = c(e) - \lambda_* + u(\text{tail}(e)) - u(\text{head}(e))$  is affine with Lipschitz constant 1 in  $\|\cdot\|_\infty$ . Hence  $|c_u(\theta, e) - c_u(\theta', e)| \leq \|c(\theta) - c(\theta')\|_\infty \leq L_c \|\theta - \theta'\|$ . The minimum of finitely many  $L_c$ -Lipschitz functions is  $L_c$ -Lipschitz, which yields the claim for  $\gamma$ .  $\square$

**Lemma 34.19** (Lipschitz constant for the Doeblin margin under template parameterization). *Assume  $\Theta$  lies in a frozen witness chamber. Fix a witness word  $w = w_1 \cdots w_L$ . Assume each template map  $\theta \mapsto A^{(k)}(\theta)$  is Lipschitz in entrywise  $\|\cdot\|_{\infty, \text{ent}}$  with constant  $L_A$ , and assume a uniform bound  $\max_{\theta \in \Theta} \max_{k, i, j} A_{ij}^{(k)}(\theta) \leq M$ . Let  $P_w(\theta) = A^{(w_L)}(\theta) \cdots A^{(w_1)}(\theta)$  and  $\delta_w(\theta) = \min_{i, j} (P_w(\theta))_{ij}$ . Then  $\delta_w$  is Lipschitz with constant*

$$L_\delta \leq L L_A M^{L-1}.$$

*Proof.* The telescoping identity gives  $P_w(\theta) - P_w(\theta') = \sum_{t=1}^L A^{(w_L)}(\theta) \cdots A^{(w_{t+1})}(\theta) (A^{(w_t)}(\theta) - A^{(w_t)}(\theta')) A^{(w_{t-1})}(\theta') \cdots A^{(w_1)}(\theta')$  with the obvious conventions at the ends. Taking entrywise  $\|\cdot\|_{\infty, \text{ent}}$  and using the uniform bound  $M$  yields  $\|P_w(\theta) - P_w(\theta')\|_{\infty, \text{ent}} \leq \sum_{t=1}^L M^{L-1} \|A^{(w_t)}(\theta) - A^{(w_t)}(\theta')\|_{\infty, \text{ent}} \leq L M^{L-1} L_A \|\theta - \theta'\|$ . Since  $\delta_w$  is the minimum of the finitely many entries of  $P_w$ , it is 1-Lipschitz as a function of the entry vector, hence it inherits the same constant.  $\square$

**Remark 34.20** (Verified bounds without smoothness). *The formulas in Lemmas 34.18 and 34.19 only use finite-dimensional Lipschitz envelopes and do not require differentiability. When a parameter map is only piecewise-Lipschitz, one may apply the same verification locally on each piece and then use the chamberwise cover mechanism (Algorithm 9).*

## 35 A Global Cover/Atlas Decider on Compact Normalized Families

The robust-certificate contract makes every local claim (tight-core structure, Doeblin persistence, recurrence mode, and rate pack) an *open* property with an explicit stability radius. This section turns that openness into a *global* statement on a fixed compact normalized family  $\Theta$  within the certified class. The outcome is a finite, witness-producing decision procedure: either the family is certified globally by a single paired robustness radius and uniform rate pack, or the procedure returns an explicit obstruction/degeneracy report identifying where and why a global certificate cannot hold.

### 35.1 Setup and the property being decided

Fix a compact parameter set  $\Theta$  together with a normalization guaranteeing uniform bounds on all template entries and reduced-data constants appearing in the robust certificate. For each  $\theta \in \Theta$ , the instance induces reduced graph data and the certificate pipeline returns exactly one of the outcomes in Section 22.

**Definition 35.1** (Good-global contraction universality on  $\Theta$ ). *We say good-global contraction universality holds on  $\Theta$  if there exist  $\epsilon_{\text{global}} = (\epsilon_{c, \text{global}}, \epsilon_{A, \text{global}})$  and a uniform rate pack  $(\tau_{\text{global}}, C_{\text{global}}, \kappa_{\text{global}})$  such that for every  $\theta \in \Theta$  the pipeline returns a contraction certificate and, moreover, the certificate remains valid under all perturbations satisfying  $\|\Delta c\| \leq \epsilon_{c, \text{global}}$  and  $\|\Delta A\| \leq \epsilon_{A, \text{global}}$  (in the chosen perturbation model), yielding uniform exponential projective contraction with the stated rate pack. We also write the conservative scalar floor  $\epsilon_{\min, \text{global}} := \min\{\epsilon_{c, \text{global}}, \epsilon_{A, \text{global}}\}$  when a single-radius parameter-space cover statement is sufficient.*

**Definition 35.2** (Boundary-global regime atlas on  $\Theta$ ). *A boundary-global regime atlas on  $\Theta$  is a finite, witness-producing report that either certifies Definition 35.1, or else exhibits explicit obstruction witnesses and/or a nonempty uncovered set together with diagnostics placing it near the named degeneracy loci  $\{\gamma = 0\} \cup \{\delta = 0\} \cup \{\text{RC1 failure}\}$ .*

The atlas viewpoint is the honest global completion. It never asserts uniform contraction on a region that intersects the degeneracy boundary, and it never leaves the exceptional set implicit.

### 35.2 Decision procedure and completeness

The proof mechanism has two ingredients already established:

- the robust certificate predicates are open with explicit stability radii (Sections 26, 27, 28, 29);

- the region-cover route: finitely many certificates cover  $\Theta$  by neighborhoods of radius  $\varepsilon_{\min}^*(\theta_i)$  (Section 33).

We now package these into a theorem whose conclusion is *algorithmic and dichotomic*.

**Theorem 35.3** (Global cover/atlas decider on  $\Theta$ ). *Assume  $\Theta$  is compact and that the normalization chosen in Section 3 provides uniform bounds sufficient to evaluate every constant appearing in the robust certificate fields. Run the region-mode procedure of Section 33 with adaptive refinement. The stopping criterion is: either the certified neighborhoods close coverage (uncovered set empty), or the remaining uncovered set is certified to lie inside the declared degeneracy neighborhood; in both cases the GlobalUniversalityReport is the output object. Then the procedure terminates and outputs a GlobalUniversalityReport matching the schema in `global_universality_report_schema.json`, with exactly one of the following outcomes.*

***the certified class**) The uncovered set is empty. Then Definition 35.1 holds on  $\Theta$  with  $\varepsilon_{\text{global}} := (\min_i \varepsilon_c^*(\theta_i), \min_i \varepsilon_A^*(\theta_i))$  and a uniform rate pack obtained by taking conservative maxima/minima over the finitely many center certificates.*

*For downstream use, this certified case can also be emitted as a standalone GlobalUniversalityCertificate object (a compressed form of the report) matching `global_universality_certificate_schema.json`.*

***dary-global atlas**) The output contains either*

- *a finite obstruction witness (one of OG1–OG4) at some sampled parameter, or*
- *a nonempty uncovered set together with diagnostics placing it inside a neighborhood of the explicit degeneracy surfaces  $\{\gamma = 0\} \cup \{\delta = 0\} \cup \{\text{RC1 failure}\}$  as defined in Section 34.*

*In this case, no choice of a single paired radius can certify the entire region by the robust-certificate method without entering the reported degeneracy neighborhood.*

*Proof.* For each parameter value  $\theta$ , the robust certificate defines an explicit paired radius  $\varepsilon^*(\theta)$  whenever the certificate branch conditions have positive margins. By the openness lemmas from the robustness layers (robust certificate contract through recurrence stability), the same certificate remains valid throughout the corresponding perturbation neighborhood. Compactness of  $\Theta$  implies there exists a finite subcover of any open cover. The region-mode algorithm either constructs such a finite subcover explicitly (yielding empty uncovered set), or it refines until any remaining uncovered region lies within the declared degeneracy neighborhood, since outside that neighborhood the margins are bounded away from zero by the Lipschitz-net verification route of Lemma 34.13. Obstruction witnesses OG1–OG4 are finite and explicit by the pipeline contract. The report combines these outputs.  $\square$

**Theorem 35.4** (Algorithmic cover/atlas guarantee with explicit net size). *Assume  $\Theta \subset \mathbb{R}^d$  is compact, normalized, and lies in a frozen witness chamber cover (Definition 34.17) so that the verification inputs  $L_\gamma$  and  $L_\delta$  are available (for example by Lemmas 34.18 and 34.19). Fix a mesh size  $h > 0$  and form an  $h$ -net  $\mathcal{N} \subset \Theta$  in  $\|\cdot\|_\infty$ . Then:*

- **(Finite work bound.)** *There exists such a net with  $|\mathcal{N}| \leq (\lceil D_\infty/h \rceil + 1)^d$  as in Lemma 34.12. Hence any non-adaptive region run that evaluates the pointwise pipeline on all  $\theta_i \in \mathcal{N}$  performs at most  $|\mathcal{N}|$  pipeline evaluations.*

- *(Correctness away from an explicit margin band.)* Let  $\underline{\gamma}_0$  and  $\underline{\delta}_0$  be the certified lower bounds produced by Proposition 34.14 on each frozen witness piece. If  $\underline{\gamma}_0 > 0$  and  $\underline{\delta}_0 > 0$  and RC1 holds for that piece, then the piece is certified good and the global uniform-margin route applies there. Otherwise the piece is certified boundary and the report may conservatively localize the remaining uncertainty to the explicit margin bands

$$\mathcal{B}_\gamma(h) := \{\theta \in \Theta : \gamma(\theta) \leq \min_{\theta_i \in \mathcal{N}} \gamma(\theta_i) + L_\gamma h\}, \quad \mathcal{B}_\delta(h) := \{\theta \in \Theta : \delta_w(\theta) \leq \min_{\theta_i \in \mathcal{N}} \delta_w(\theta_i) + L_\delta h\},$$

and, when applicable, the discrete RC1/OG4 label determined by the avoidance automaton (Lemma 34.15). In particular, the complement  $\Theta \setminus (\mathcal{B}_\gamma(h) \cup \mathcal{B}_\delta(h))$  is certified to lie in the regime where both margins are bounded below by the net-certified values.

Consequently, combining the piecewise outcomes produces a finite GlobalUniversalityReport in the sense of Definition 35.2 whose certified pieces come with explicit radii and rate packs, while any non-certified remainder is explicitly labeled by margin-band diagnostics.

*Proof.* The existence and size bound for  $\mathcal{N}$  is Lemma 34.12. On each frozen witness piece, Proposition 34.14 supplies certified lower bounds  $\underline{\gamma}_0$  and  $\underline{\delta}_0$  from the net minima and the Lipschitz envelope. If both bounds are strictly positive and RC1 holds, then the uniform-margin hypotheses hold on that piece and the good-global contraction conclusions follow by Theorem 34.2. If one of the bounds fails to be strictly positive, then the Lipschitz envelope still implies that any point where the corresponding margin is large must lie outside the explicit band set defined by the net minima plus  $Lh$ . The RC1/OG4 label is determined by cycle detection in the avoidance automaton and is constant on a frozen witness piece. Assembling the finitely many piecewise conclusions yields the stated global report.  $\square$

**Remark 35.5** (Certificate object for downstream use). *When the outcome is certified, the global information needed by a third party is typically smaller than the full region report. The file `global_universality_certificate_schema.json` records a minimal object containing:  $\epsilon_{\text{global}}$ ,  $\gamma_0$ ,  $\delta_0$ , a witness record (uniform or chamberwise), and the uniform rate pack. This is the natural carry-forward artifact for later proofs or numerical experiments.*

**Corollary 35.6** (When the decider must certify: degeneracy avoidance on finitely many witness chambers). *Assume  $\Theta$  is compact and is covered by finitely many frozen witness chambers in the sense of Definition 34.17. If on each chamber the degeneracy sets of Definition 34.3 are empty (equivalently: the verified lower bounds in Proposition 34.14 are strictly positive and RC1 holds on that chamber), then the region-mode procedure returns the certified outcome of Theorem 35.3. In particular, good-global contraction universality holds on  $\Theta$  with an explicit uniform paired radius  $\epsilon_{\text{global}} := \min_{\text{chambers } \Omega} \epsilon_{\text{global}}(\Omega)$  (componentwise minimum) and a uniform rate pack.*

*Proof.* On each frozen witness chamber  $\Omega$ , the emptiness of degeneracy sets implies uniform robust margins by Proposition 34.4, and the explicit net+Lipschitz route supplies certified lower bounds when desired (Proposition 34.14). Thus Theorem 34.2 applies chamberwise and produces an explicit chamberwise paired radius  $\epsilon_{\text{global}}(\Omega)$ . Finiteness of the chamber cover gives a positive global componentwise minimum, and compactness yields a finite cover by robust certificate neighborhoods. Therefore the region-mode decider closes coverage and outputs the certified case.  $\square$

### 35.3 Consequences for decidability islands

Theorem 35.3 converts a qualitative “robustness in a neighborhood” statement into a concrete, checkable conclusion for parameter families. In particular, it yields a sharp *decidable island* statement for robust switching stability: on the certified positive/tight/Doebelin subclass supporting the

pipeline, the question “does this compact normalized family admit a uniform robust contraction mechanism?” reduces to a finite cover computation plus explicit degeneracy diagnostics.

**Remark 35.7** (What this does and does not claim). *The decider above is a family-level completeness statement for the certified class. It does not claim that good-global contraction universality holds for all possible parameter families, nor that the general joint spectral radius decision problem is decidable. Instead, it provides an explicit frontier: inside the certified class and under normalization, the program decides the outcome by producing either a global certificate or explicit obstruction/degeneracy data.*

### 35.4 Artifact-level completion: report, certificate, and paired radii

The mathematics above naturally produces two levels of machine-checkable output. The intent is that an independent reader can: (i) validate the global conclusion without rerunning the full internal pipeline, and (ii) carry forward the minimal global data needed to support later proofs.

#### Paired radii and the conservative scalar floor

The robust layer distinguishes perturbations of costs and perturbations of template entries. Accordingly, the canonical robustness radius is a pair  $\varepsilon = (\varepsilon_c, \varepsilon_A)$  with the meaning  $\|\Delta c\| \leq \varepsilon_c$  and  $\|\Delta A\| \leq \varepsilon_A$ . For convenience, we also record the conservative scalar floor  $\varepsilon_{\min} := \min\{\varepsilon_c, \varepsilon_A\}$ . The scalar floor is sufficient for parameter-space cover arguments; the pair is the natural object for reproducible robustness claims.

#### Report versus certificate

| Artifact                                   | Role                                                                   | When produced                    |
|--------------------------------------------|------------------------------------------------------------------------|----------------------------------|
| <code>GlobalUniversalityReport</code>      | Full region-level audit: cover data, diagnostics, and optional proofs  | Always (certified or boundary)   |
| <code>GlobalUniversalityCertificate</code> | Compressed carry-forward object: global bounds + witnesses + rate pack | Only when uncovered set is empty |

Both artifacts are explicitly specified by JSON Schemas:

- `global_universality_report_schema.json`
- `global_universality_certificate_schema.json`

**FrontierReport and FrontierAtlasReport.** To make the “frontier atlas” concept an exported object rather than an internal metaphor, the repository also includes a compact instance-level *FrontierReport* and a region-level *FrontierAtlasReport* schema. The intent is that a reader can carry a minimal, standardized summary even when a full global audit report is not required. These objects are specified by:

- `frontier_report_schema.json` (see `frontier_report_example.json`)
- `frontier_atlas_report_schema.json` (see `frontier_atlas_report_example.json`)

In implementations, `GlobalUniversalityReport` can be treated as the detailed audit object, while `FrontierAtlasReport` is the standardized chamberwise summary (with explicit declared loci and localization statements).

The report schema supports the paired-radius form by allowing `epsilon_global` and `epsilon_star` to be either a scalar or an object with fields `eps_c_star`, `eps_A_star`, and (optionally) `eps_min`. The examples in this repository use the paired form.

### Compression map: `report` $\rightarrow$ `certificate`

In the certified case, the report contains more information than downstream use typically requires. The compression map discards only: coverage bookkeeping (net points and intermediate certificate IDs), optional Lipschitz proof blocks, and free-text diagnostics. It retains:

- global margins  $(\gamma_0, \delta_0)$ ,
- the paired global robustness radius  $\epsilon_{\text{global}}$ ,
- the witness payload (uniform or chamberwise),
- and the uniform rate pack.

This is the exact data needed to reuse the global conclusion as an input to later arguments.

### Schema validation

The schema files are included so that the report and certificate objects can be validated independently with any standards-compliant JSON Schema validator.

### Machine description of the target family $\Theta$

The global decider is a statement on a *region*, so the artifacts include a separate, explicit descriptor for the parameter family. The schema `theta_family_schema.json` defines a compact, model-agnostic format for declaring a target region (box/simplex/curve) together with a normalization choice. The file `theta_family_example.json` records the canonical family  $\Theta_{\text{can}}$  from Section 5.

When one invokes the “global-on- $\Theta$  by net + Lipschitz envelope” route (Proposition 34.14), the descriptor also makes the required continuity inputs explicit via the `net_certification_contract` block. This block declares the parameter norm and conservative moduli  $(L_c, L_A, M)$  used to certify uniform positive margins from finite sampling.

To make the two global outcomes concrete, the repository also includes:

- `global_universality_report.ThetaCan_certified.json` (certified-good-global format example)
- `global_universality_certificate.ThetaCan.json` (compressed carry-forward certificate)
- `global_universality_report.ThetaCan_atlas.json` (boundary-global atlas format example)

## 36 Sharpness and Realizability of the Obstruction Taxonomy

This section records explicit, finite examples showing that each obstruction class OG1–OG4 is *genuinely necessary*. We also give a concrete “good regime” example where OG1–OG4 do not occur and the forcing mechanism (Sections 10 and 12) yields a nonvacuous unconditional contraction rate.

Throughout, we present examples at the reduced-graph/template level:

- vertices represent descriptor states,
- each directed edge  $e : u \rightarrow v$  carries a nonnegative template matrix  $T_e$ ,
- the canonical support sets  $S(\cdot)$  are specified explicitly and checked by support propagation (Definition 13.1).

**Proposition 36.1** (Necessity of the obstruction taxonomy). *No obstruction class OG1–OG4 can be removed from the witness dichotomy without losing correctness. More precisely, for each  $i \in \{1, 2, 3, 4\}$  there exists a finite reduced instance for which:*

- all obstruction checks OG $j$  with  $j < i$  fail (so the instance passes every earlier gate),
- obstruction OG $i$  holds, and
- no uniform gap-free contraction conclusion of the form stated in the main theorem chain can be justified without detecting OG $i$ .

*Proof.* The explicit constructions in the subsections below witness this gate-by-gate necessity. The OG1 example has no aperiodic tight SCC, so no Doeblin regime is available. The OG2 example has a tight SCC but violates face-map consistency, so any conclusion that assumes a coherent active face is invalid. The OG3 examples satisfy OG1 and OG2 but admit no Doeblin word on the active face, so the Birkhoff/Doeblin contraction route cannot start. The OG4 example satisfies OG1–OG3 and contains a Doeblin word, yet that word is avoidable along a periodic tight trajectory; without detecting this avoidance cycle, one would incorrectly assert a uniform recurrence-driven contraction rate.  $\square$

### 36.1 OG1 is realizable (no aperiodic tight SCC)

Let  $V = \{a, b\}$  and let the tight graph have exactly the edges  $a \rightarrow b$  and  $b \rightarrow a$ . This SCC has period 2, hence it is imprimitive. Set  $d = 1$  and take  $T_{a \rightarrow b} = T_{b \rightarrow a} = [1]$ . Then every edge is tight, the unique SCC has period 2, and therefore OG1 holds. No aperiodic tight SCC exists, so no Doeblin/minorization regime can be entered.

### 36.2 OG2 is realizable (face-map inconsistency cycle)

Let  $V = \{u, v\}$ ,  $d = 2$ . Define support sets

$$S(u) = \{1\}, \quad S(v) = \{1\}.$$

Let the reduced graph contain both edges  $u \rightarrow v$  and  $v \rightarrow u$  (so it is an SCC). Assign the template on the edge  $e : u \rightarrow v$  by

$$T_e = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then for  $x \in \mathbb{R}_{>0}^{S(u)}$  we have  $T_e x = (x_1, x_1)$ , so coordinate 2 is activated. But  $2 \notin S(v)$ , hence the “no spurious activation” condition in Definition 13.1 fails. By Lemma 13.9, the edge  $e$  together with the marked indices  $(i, j) = (2, 1)$  and any return path  $v \rightsquigarrow u$  produces a finite inconsistency cycle witness. Thus OG2 is realizable.

### 36.3 OG3 is realizable (no Doeblin word on the active face)

We record two standard sharp shapes corresponding to the two OG3 witness types.

#### 36.3.1 OG3(cut) realizable: block-triangular support

Let  $V = \{h\}$  (one-vertex SCC),  $d = 2$ ,  $S(h) = \{1, 2\}$ . Let the unique edge be a self-loop  $e : h \rightarrow h$  with

$$T_e = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then  $S(\cdot)$  is a face map and  $S(H) = \{1, 2\}$ . However, there is no arrow  $2 \rightarrow 1$  in the coordinate digraph  $\Gamma(H)$  (Definition 13.4), so  $\Gamma(H)$  is not strongly connected. Equivalently, letting  $A = \{1\}$  and  $B = \{2\}$ , we have  $(T_e)_{1,2} = 0$  and hence  $(B(w))_{1,2} = 0$  for every word  $w$ . Thus no word can be strictly positive on  $S(H) \times S(H)$  and OG3 holds with a cut witness (Lemma 13.10).

#### 36.3.2 OG3(product-automaton) realizable: permutation-only regime

Let  $V = \{h\}$ ,  $d = 2$ ,  $S(h) = \{1, 2\}$ . Let the allowed generator templates be the two permutation matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let the reduced graph have two self-loop edges at  $h$  labeled by  $P$  and  $Q$ . Then  $\Gamma(H)$  is strongly connected (each coordinate reaches the other under  $Q$ ), but every product is still a permutation matrix, so no word yields strict positivity on  $S(H) \times S(H)$ . This is exactly the “strongly connected but not Doeblin” counter-shape in Remark 13.7, and the Boolean-pattern product automaton returns a finite non-reachability certificate. Thus OG3 is realizable even in the presence of strong connectivity.

### 36.4 OG4 is realizable (Doeblin exists but is avoidable)

We now realize OG4 in the regime where OG1–OG3 fail. Let  $V = \{h, x, y\}$  and consider the tight SCC  $H$  supported on these vertices. Take  $d = 2$  and  $S(v) = \{1, 2\}$  for all  $v$ . Let there be edges

$$h \rightarrow x, \quad x \rightarrow h, \quad x \rightarrow y, \quad y \rightarrow x.$$

This SCC is aperiodic (it contains a 2-cycle  $h \rightarrow x \rightarrow h$  and a 3-cycle  $x \rightarrow y \rightarrow x \rightarrow h \rightarrow x$ ). Assign templates:

$$T_{h \rightarrow x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (\text{strictly positive}), \quad T_{x \rightarrow h} = I, \quad T_{x \rightarrow y} = Q, \quad T_{y \rightarrow x} = Q,$$

where  $I$  is the identity and  $Q$  is the swap matrix from the preceding example. Then the single-edge word  $w = (h \rightarrow x)$  is a Doeblin word (already strictly positive), so OG3 fails. However, the periodic tight path alternating  $x \rightarrow y \rightarrow x \rightarrow y \rightarrow \dots$  avoids the vertex  $h$  entirely, so it avoids

$w$  entirely. In particular, the avoidance graph  $\mathcal{A}_{\neg w}(H)$  has a directed cycle and  $\text{OG4}(H, w)$  holds (Definition 18.1).

This shows that even in the “good” structural regime where a Doeblin word exists, bounded-gap recurrence of that *chosen* word is not automatic.

### 36.5 Nonvacuous good regime: OG1–OG4 fail and forcing succeeds

Finally, we give a compact example where the forcing mechanism is nontrivial. Let  $V = \{h, x, y\}$  and edges

$$h \rightarrow x, \quad x \rightarrow h, \quad x \rightarrow y, \quad y \rightarrow h.$$

Assume these are the only outgoing edges from  $h$  (so  $h$  has outdegree 1). The SCC  $H$  on  $\{h, x, y\}$  is strongly connected and aperiodic: there is a cycle of length 2 ( $h \rightarrow x \rightarrow h$ ) and a cycle of length 3 ( $h \rightarrow x \rightarrow y \rightarrow h$ ). Let  $d = 2$  and  $S(v) = \{1, 2\}$  for all  $v$ . Assign templates

$$T_{h \rightarrow x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_{x \rightarrow h} = I, \quad T_{x \rightarrow y} = I, \quad T_{y \rightarrow h} = I.$$

Then  $w = (h \rightarrow x)$  is a Doeblin word. Moreover, because  $h$  has a unique outgoing edge, *every* infinite path in  $H$  contains the edge  $h \rightarrow x$  with gaps at most 3 (in fact at most the maximum return time to  $h$ ). Equivalently, the avoidance graph  $\mathcal{A}_{\neg w}(H)$  is acyclic, so RC1 holds and OG4 fails. Therefore the forcing dichotomy applies (Corollary 12.5), and Theorem 19.7 yields a uniform exponential contraction rate on all tight minimizers supported in  $H$ .

**Remark 36.2** (What this demonstrates). *This example is intentionally simple: the Doeblin block is a single positive generator and recurrence is forced by a finite outdegree constraint. The point is not complexity but nonvacuity: the “OG1–OG4 fail  $\Rightarrow$  contraction” regime is populated, and the forcing lemmas deliver explicit rates from finite witnesses.*

## 37 Minimality and sharpness: why the boundary is tight

This section records short “edge-case” constructions that explain why the hypotheses and obstruction gates in the boundary theorem cannot be substantially weakened without changing the conclusion. The goal is not to catalog counterexamples, but to isolate the minimal mechanisms that force the regime split.

### 37.1 Why tight-core reduction is not optional

The calibration potential  $\phi$  and the induced tight graph  $G^{(0)}$  are not merely a technical convenience. They are the device that makes the extremal problem finite and local. Without passing to tight edges, statements about minimizers are typically false if one reads them directly on the unreduced descriptor graph.

**Remark 37.1** (A two-cycle tie shows why one must reduce). *Consider a finite graph with two disjoint cycles  $\gamma_1$  and  $\gamma_2$  that have the same cycle mean  $\lambda_*$ . Assume there are additional edges of slightly higher mean that connect the cycles in both directions. Then minimizing measures may be supported on either cycle, or on mixtures that switch between the two cycles with sublinear frequency. On the unreduced graph there is no canonical “core” and no unique active face. After calibration, the tight subgraph separates the true minimizer-supporting structure from slack edges; the subsequent obstruction tests are meaningful only on this tight reduction.*

## 37.2 How contraction fails: realizable obstructions

The obstruction gates OG2 and OG3 are designed to be witness-producing and to capture genuinely different failure modes. Both occur in simple finite-template systems.

- **OG2 (face-map inconsistency) is a real obstruction.** It is possible for tight edges to force incompatible coordinate support requirements. Concretely, one can have two tight words  $u$  and  $v$  in the same tight SCC such that  $u$  maps mass from an active face into a strict subface, while  $v$  maps mass into a different strict subface, and the SCC allows alternating concatenations. The witness produced by Lemma `lem:og2-witness-cycle` is exactly the finite inconsistency cycle that certifies that no consistent face map exists.
- **OG3 (no Doeblin word) is strictly stronger than “ $\Gamma(H)$  not strongly connected.”** Even when the union support digraph  $\Gamma(H)$  is strongly connected, the allowed face-restricted patterns may generate a semigroup that never contains an all-positive block. The clean counter-shape is the “permutation-pattern” family: if the admissible patterns on the face are permutation matrices that generate a transitive action, then  $\Gamma(H)$  is strongly connected but every product remains a permutation pattern, so no Doeblin word exists. This is why the certificate used in the paper is the Boolean-pattern product-automaton test: Doeblin existence is equivalent to reachability of the all-ones pattern state.

## 37.3 Why Doeblin existence alone does not force contraction on minimizers

The preceding discussion explains OG2 and OG3. There is a further sharp obstruction that is easy to underestimate: even when a Doeblin word *exists* on an aperiodic tight SCC, it may be *avoidable* along some tight trajectories. If the cost normalization ties all tight edges (zero reduced cost), then such avoidable tight trajectories are minimizing. This is exactly the role of OG4 and the recurrence certificate layer (RC1).

**Proposition 37.2** (OG4 is a genuine minimizer-level obstruction). *There exists a finite reduced descriptor instance for which:*

- *the tight core contains an aperiodic tight SCC  $H$  with a consistent active face  $S(H)$ ,*
- *a Doeblin word  $w$  supported in  $H$  exists (indeed, a single tight edge is strictly positive on  $S(H) \times S(H)$ ),*
- *yet there is a minimizing periodic tight trajectory whose edge itinerary avoids  $w$  entirely,*
- *and along that minimizing trajectory there is no projective contraction on  $\mathbb{R}_{>0}^{S(H)}$ .*

*Consequently, no theorem of the form “Doeblin exists on  $H \Rightarrow$  uniform contraction on all minimizers supported in  $H$ ” can hold without an additional recurrence/forcing hypothesis.*

*Proof.* Use the explicit OG4 realization from Section 36. Let  $H$  have vertices  $\{h, x, y\}$  and tight edges  $h \rightarrow x$ ,  $x \rightarrow h$ ,  $x \rightarrow y$ ,  $y \rightarrow x$ . Let  $d = 2$  and  $S(v) = \{1, 2\}$  for all  $v$ . Assign templates

$$T_{h \rightarrow x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_{x \rightarrow h} = I, \quad T_{x \rightarrow y} = Q, \quad T_{y \rightarrow x} = Q,$$

where  $I$  is the identity and  $Q$  is the swap (permutation) matrix. Declare every edge to have reduced cost 0, so every bi-infinite tight path in  $H$  is minimizing.

Then the single-edge word  $w = (h \rightarrow x)$  is Doeblin because  $T_{h \rightarrow x}$  is strictly positive. However, the periodic itinerary  $\pi = (x \rightarrow y \rightarrow x \rightarrow y \rightarrow \dots)$  avoids the vertex  $h$  and hence avoids  $w$  entirely. Along this trajectory, the cocycle is a product of permutation matrices (powers of  $Q$ ), so the projective action is an isometry on the positive cone: for all  $u, v \in \mathbb{R}_{>0}^2$ ,  $d_H(Q^n u, Q^n v) = d_H(u, v)$ . Thus there is no exponential contraction along this minimizing orbit.

This is precisely the OG4 mechanism: Doeblin exists on  $H$  but is avoidable, so without RC1 (or an alternative forcing mechanism) uniform contraction on all minimizers cannot be concluded.  $\square$

**Remark 37.3** (Interpretation for the gap-free upgrade route). *Proposition 37.2 explains why the paper separates existence of a Doeblin word from recurrence of that word on minimizing dynamics. The obstruction is not analytic subtlety; it is a finite, combinatorial avoidance phenomenon. Accordingly, the correct “upgrade” object is a witness-producing recurrence certificate (RC1) or a forcing lemma that guarantees positive density of  $w$  on the trajectories of interest.*

### 37.4 When imprimitive trapping is the right conclusion

The regime split treats aperiodicity (period 1) and imprimitivity (period  $p > 1$ ) differently because the projective geometry is different. In an imprimitive tight SCC, the index set splits into cyclic classes and products respect that cyclic structure. Even with strong tightness and perfect calibration, one should not expect a single stationary projective section on the full face.

**Remark 37.4** (Imprimitive tight cores can force cycling). *Let  $H$  be a tight SCC with period  $p > 1$ . Then there is a cyclic decomposition of states into classes  $C_0, \dots, C_{p-1}$  such that every edge advances the class modulo  $p$ . Along an admissible tight trajectory, the support of mass on the active face can rotate among these classes. This produces a genuine periodic trapping phenomenon at the level of projective directions. Accordingly, the boundary theorem outputs “imprimitive trapping” (periodic regime) rather than uniform contraction on the full face.*

Taken together, these examples explain the sharpness of the theorem chain: the tight-core reduction is needed to make minimizers finite and local; OG2 and OG3 isolate distinct algebraic failures on the active face; and the imprimitive branch is the correct endpoint when the tight core has period  $> 1$ .

## 38 Parameter Chambers and Generic Cycle Selection

This section formalizes what is generic in linear parameter families in a literal, checkable sense: uniqueness of the minimizing cycle mean off a finite hyperplane arrangement, together with a finite chamber decomposition of parameter space on which the minimizing set is constant. No claim of “generic aperiodicity” is made, since periodic/aperiodic is a combinatorial property of the selected critical SCC.

### 38.1 Linear parameter model and hyperplane arrangement

Fix a finite directed graph  $G = (V, E)$  (after reduction) and a feature map  $\psi : E \rightarrow \mathbb{Z}^k$ . For  $\theta \in \mathbb{R}^k$  define edge costs

$$c_\theta(e) = \langle \theta, \psi(e) \rangle,$$

and for a directed cycle  $\gamma$  define its mean cost

$$\mu_\theta(\gamma) := \frac{1}{|\gamma|} \sum_{e \in \gamma} c_\theta(e).$$

Let  $\mathcal{C}$  be the finite set of simple directed cycles in  $G$  and let

$$\lambda_*(\theta) := \min_{\gamma \in \mathcal{C}} \mu_\theta(\gamma)$$

be the minimal cycle mean.

**Definition 38.1** (Degeneracy hyperplanes). *For distinct cycles  $\gamma \neq \gamma'$  define the affine functional*

$$F_{\gamma, \gamma'}(\theta) := \mu_\theta(\gamma) - \mu_\theta(\gamma').$$

*The degeneracy arrangement is the finite union of hyperplanes*

$$\mathcal{D} := \bigcup_{\gamma \neq \gamma'} \{\theta : F_{\gamma, \gamma'}(\theta) = 0\}.$$

**Lemma 38.2** (Unique minimizing cycle off  $\mathcal{D}$ ). *If  $\theta \notin \mathcal{D}$ , then there exists a unique cycle  $\gamma_*(\theta) \in \mathcal{C}$  with  $\mu_\theta(\gamma_*(\theta)) = \lambda_*(\theta)$ .*

*Proof.* If two distinct cycles both attained the minimum, then  $F_{\gamma, \gamma'}(\theta) = 0$  for that pair, so  $\theta \in \mathcal{D}$ .  $\square$

## 38.2 Chamber decomposition and critical SCC stability

**Definition 38.3** (Chambers). *A parameter chamber is a connected component of  $\mathbb{R}^k \setminus \mathcal{D}$ .*

**Proposition 38.4** (Finite chamber structure).  *$\mathbb{R}^k \setminus \mathcal{D}$  is a finite union of open polyhedral chambers. On each chamber, the minimizing cycle  $\gamma_*(\theta)$  is constant.*

*Proof.*  $\mathcal{D}$  is a finite arrangement of affine hyperplanes, so its complement has finitely many connected components, each described by a finite system of strict linear inequalities  $F_{\gamma, \gamma'}(\theta) \leq 0$ . Within a fixed component, the sign pattern of all  $F_{\gamma, \gamma'}$  is constant, hence the unique minimizer  $\gamma_*$  is constant by Lemma 38.2.  $\square$

**Corollary 38.5** (Computable periodic/aperiodic regime on chambers). *Fix a chamber  $\mathcal{U}$  and let  $\gamma_*$  be its minimizing cycle. Then the essential critical SCC of the tight graph is the SCC generated by  $\gamma_*$  (in particular, it is periodic). More generally, on lower-dimensional strata  $\theta \in \mathcal{D}$  where multiple cycles tie, the tight critical subgraph may contain multiple cycles. It may be periodic or aperiodic depending on the gcd of cycle lengths in the resulting critical SCC. All of these properties are decidable from the finite tight geometry.*

**Theorem 38.6** (Boundary type is constant on chambers). *Fix the graph  $G$  and feature map  $\psi$  as above, and consider the induced linear cost family  $c_\theta$ . For each chamber  $\mathcal{U}$  of  $\mathbb{R}^k \setminus \mathcal{D}$ , the following data are constant over  $\theta \in \mathcal{U}$ :*

- the minimizing cycle  $\gamma_*$  and minimal mean  $\lambda_*(\theta)$  (hence the critical/tight SCC);
- the periodic/aperiodic regime of the essential critical SCC;
- the boundary classification output of the Tight-Core Boundary Theorem on that SCC (imprimitive trapping, or in the aperiodic case: certificate/Doeblin/contraction versus OG2/OG3 obstructions, together with the associated witness objects).

*Consequently, the boundary type can change only when crossing the degeneracy arrangement  $\mathcal{D}$ .*

*Proof.* On a chamber  $\mathcal{U}$  the sign pattern of all cycle-mean differences  $F_{\gamma, \gamma'}$  is fixed, so the minimizer  $\gamma_*$  is constant by Proposition 38.4. In the unique-minimizer case, the critical/tight subgraph is generated by the edges of  $\gamma_*$ , hence the essential critical SCC (and its periodicity) is constant. All subsequent boundary outputs in the Tight-Core Boundary Theorem are computed from finite graph data on the essential SCC and its induced tight geometry (certificate graph construction, SCC type, and the witness-producing obstruction tests), so they are constant once that finite geometry is fixed. Crossing a hyperplane in  $\mathcal{D}$  is the only way the minimizer set can change, hence it is the only place where the induced tight/critical geometry and boundary type can change.  $\square$

**Edge cases on strata  $\theta \in \mathcal{D}$ .** On the degeneracy arrangement, ties produce a critical subgraph with multiple cycles. The same finite pipeline applies, but the tight geometry must be formed from the tied minimizing set. For convenience, the most common tie patterns can be summarized as follows.

| Situation on $\mathcal{D}$                       | What is decidable (and what to report)                                                                                                                                                                                                                |
|--------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Ties among cycles <i>within one</i> critical SCC | Compute the gcd of cycle lengths in that SCC. If the gcd is $> 1$ , the regime is periodic (trapping). If the gcd is 1, the SCC is aperiodic and the Tight-Core Boundary Theorem applies (certificate/Doeblin/contraction or OG2/OG3 with witnesses). |
| Ties split across <i>multiple</i> critical SCCs  | The tight/critical graph decomposes into SCCs. Run the boundary classification on each essential SCC and report the resulting regime/witnesses; the overall boundary type is the finite list of SCC outcomes.                                         |
| Higher-codimension ties (many cycles)            | Form the tight subgraph induced by the minimizing set and apply the same SCC + certificate/obstruction pipeline. The output remains finite and witness-producing.                                                                                     |

**Remark 38.7** (What this does (and does not) justify). *In linear cost families, uniqueness of the minimizing cycle is generic in the precise sense “off  $\mathcal{D}$ .” That generic situation typically yields a periodic critical SCC. Accordingly, any claim of “generic aperiodicity” would require additional structure beyond the linear cycle-mean model. The robust message supported here is different: the periodic/aperiodic and certificate/obstruction boundary is finitely computable from the tight geometry, and by Theorem 38.6 the boundary type is constant on chambers and can change only across  $\mathcal{D}$ .*

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| Open-problem family                                                         | Resolved statement in the certified class                                                                                                                                                                                                                                                                                                                                                                                                                       | Pipeline artifact                                            |
|-----------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------|
| Černý-type synchronizing bounds for induced automata [6, 7]                 | For the specific automata generated by the certificate program (Doeblin-word search and avoidance automata), the pipeline provides explicit state counts and either a bounded-gap modulus $B$ (acyclic avoidance) or an explicit cycle witness. When the induced automata satisfy aperiodicity conditions that are checkable from the transition monoid, known special-class synchronizing bounds apply, yielding a quantitative reset bound for this subclass. | Automaton model<br>+ modulus $B$<br>(+ optional reset bound) |
| Quantitative primitivity / length of a positive (Doeblin) product [8]       | The Doeblin-word search is finite and yields either a positive product witness with margin $\delta$ or a certificate of failure within the searched regime. In the certified setting, digraph exponent bounds and state-size bounds convert this into an explicit worst-case search radius for positivity on the core block.                                                                                                                                    | Doeblin witness word<br>(length $L$ , margin $\delta$ )      |
| Barabanov/extremal norm geometry and constructive computation [9]           | A calibrated projective section together with recurrence (RC1) yields an explicit constructive extremal geometry: a computable contraction mechanism and robustness neighborhood for the extremal dynamics. This provides a concrete, checkable analogue of an extremal-norm package on the certified class.                                                                                                                                                    | Rate pack<br>+ calibrated section<br>+ robustness radii      |
| Matrix-semigroup decision problems under bounded-language restrictions [10] | The certificate program restricts products to a constrained language (tight-core admissible words plus avoidance constraints). Within that restricted language, the key decision questions needed for stability reduce to finite automata and finite witnesses, giving an explicit decidability template compatible with bounded-language decidability phenomena.                                                                                               | Language model<br>(tight-core + constraints)                 |
| Mapping undecidable vs. decidable frontiers for matrix products [1, 3]      | The robust certificate contract isolates a decidable island and produces explicit exceptional sets where margins vanish. This provides an operational frontier map: the pipeline states exactly which failures prevent certification and returns corresponding finite witnesses.                                                                                                                                                                                | Frontier report<br>(OG / RU margins)                         |

Table 3: How the robust certificate program resolves several open-problem families within the certified positive-template class (part 2).

| Chaos-theory open-problem family                                                                                                 | Resolved statement in the certified class                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | Pipeline artifact             |
|----------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------|
| Effective description of chaos and turbulence [5]<br><br>+ robust certificate<br>+ cover/atlas report                            | The long-run extremal dynamics admit an <i>effective finite description</i> : a tight-core symbolic model (graph/SFT/sofic refinement) together with a certificate dichotomy that returns either explicit uniform projective contraction data (rates and constants) or a finite obstruction witness. On compact normalized parameter families within the certified regime, this upgrades to a <i>finite cover/atlas</i> : either a uniform good-global contraction certificate or a boundary-atlas report localizing degeneracy. | Tight-core + automata         |
| Rough vs. sensitive dependence on initial data [5]<br><br>+ margins $(\gamma, \delta, \varepsilon^*)$<br>+ obstruction witnesses | In the RU-Stable regime, extremal projective directions forget initial data at an explicit exponential rate (a quantitative “no-rough-dependence” statement on the certified observables). Outside that regime, unpredictability is localized to explicit margin-collapse loci (e.g. $\gamma \rightarrow 0$ or $\delta \rightarrow 0$ ) with finite witnesses explaining the failure mechanism.                                                                                                                                  | Rate pack $(\tau, C, \kappa)$ |
| Arrow of time / macroscopic irreversibility in chaotic settings [5]<br><br>+ recurrence modulus<br>+ obstruction taxonomy        | Uniform projective contraction on the extremal set yields a one-way “loss of memory” principle for coarse projective observables: forward iteration collapses uncertainty to a unique attracting section at an explicit rate, while obstruction regimes certify persistent memory via periodic/avoidance witnesses.                                                                                                                                                                                                              | Contraction theorem           |
| Control-paradox themes (enrichment/pesticides/plankton as timing-sensitive forcing) [5]<br><br>+ avoidance-cycle witness (OG4)   | The certificate contract isolates when forcing/switching yields genuine mixing on the active face (Doebelin + recurrence) versus when timing constraints permit long avoidance of mixing. In the latter case, the pipeline returns explicit avoidance-cycle witnesses, giving a rigorous “backfire” mechanism in the certified switching-template abstraction.                                                                                                                                                                   | Doebelin-word witness         |

Table 4: How the robust certificate program interfaces with major chaos-theory open-problem families (after Li) within the certified positive-template class.