

Extremal Rigidity for Finite-Template Cone-Preserving Cocycles

Paper 1 (Rigidity Module): a plug-compatible extremal theorem

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Abstract

We present a rigidity module for finite-template, cone-preserving matrix cocycles over a transitive SFT/sofic shift. Under **FTG** (finite templates with uniform coefficient bounds) and a **strict cycle-mean gap (SCG)** selecting a unique minimizing primitive periodic orbit w_* , plus **core strong connectivity + aperiodicity** (hence primitivity of the core product), we obtain an explicit Doeblin minorization block $B_* > 0$ and an explicit Hilbert-metric contraction factor $\tau_* < 1$.

Two results are proved: (i) *Periodic-core exact reduction*: on the nucleus orbit $\mathcal{O}_*$ we build a calibrated projective section and an exact multiplicative-additive reduction. (ii) *Nonperiodic extension (conditional)*: assuming the interface hypothesis **(PO)** (periodic realization of the global minimum), we prove a return-density lemma: every ergodic minimizing measure assigns positive mass to the Doeblin-block cylinder, hence sees infinitely many Doeblin contractions with positive frequency. This yields projective synchronization (almost surely) and a calibrated projective section on a full-measure minimizing set. The paper is intentionally modular: it proves the cone/projective rigidity implications once its interface hypotheses are supplied (by Paper 2).

1 Context and related work

The technical inputs used here are standard but appear across several literatures. Projective/Hilbert-metric contraction for positive operators originates with Birkhoff's theorem and subsequent refinements [1, 2, 3]. The subadditive growth viewpoint for matrix cocycles is governed by Kingman's subadditive ergodic theorem [4], with classical foundations for random products in [5, 6]. Our base dynamics is symbolic (SFT/sofic), for which standard references include [7].

The broader motivation (handled in Paper 2) treats finite templates as a transfer-matrix envelope for local constraints, linking to the transfer-matrix tradition in statistical mechanics [8, 9]. Finally, the strict cycle-mean gap and extremal selection sit adjacent to the joint/lower spectral radius literature [10, 11], though we do not attempt a classification result there.

2 Structural core and extremal selection

This pass front-loads the exact structural ingredients and selects the minimizing nucleus. No contraction, no minorization, and no reduction language is used here.

2.1 Model class: Finite-Template Growth (FTG)

Definition 1 (Finite-Template Growth class (FTG)). *Let $K = \mathbb{R}_{\geq 0}^d$ and let (Σ, σ) be a subshift of finite type on a finite alphabet $\mathcal{S}$. A locally constant cocycle $A : \Sigma \rightarrow \mathbb{R}_{\geq 0}^{d \times d}$, $A(x) = A_{x_0}$, belongs to*

FTG if there exist a finite template set $\mathcal{T} = \{A_s : s \in \mathcal{S}\}$ and constants $0 < \underline{a} \leq \bar{a} < \infty$ such that:

- **Finite template set.** $\mathcal{T}$ is finite and $A(x) \in \mathcal{T}$ for all $x \in \Sigma$.
- **Uniform coefficient bounds.** Every positive entry of every $T \in \mathcal{T}$ lies in $[\underline{a}, \bar{a}]$ (entries may also be 0).
- **Interior invariance (no zero rows).** Every $T \in \mathcal{T}$ has no zero rows (equivalently $T\mathbf{1} > 0$), hence $T(\text{int}(K)) \subset \text{int}(K)$. This ensures Hilbert's metric and projective normalization are defined at every step.
- **Cone preservation.** $TK \subset K$ for all $T \in \mathcal{T}$ (automatic here).
- **No state-dependent rescaling.** Template coefficients are fixed and do not depend on state magnitude.

Remark 1. FTG is a structural class. It does *not* assume primitivity, Doeblin minorization, projective contraction, or any global contraction hypothesis.

2.2 Cycle means and strict cycle-mean gap

For any admissible periodic word $w = w_0 \dots w_{n-1}$ (so $w^\infty \in \Sigma$), define

$$P(w) = A_{w_{n-1}} \cdots A_{w_0}, \quad \lambda(w) = \frac{1}{n} \log \rho(P(w)),$$

where $\rho(\cdot)$ is spectral radius.

Convention (words vs orbits). A *primitive periodic orbit* $\mathcal{O}$ is represented by any primitive word w generating it; cyclic shifts represent the same orbit and powers w^k represent the same orbit. When we say “unique minimizing nucleus”, uniqueness is always taken at the level of primitive periodic orbits.

Define

$$\lambda_* := \inf_{\mathcal{O} \text{ primitive periodic orbit}} \lambda(\mathcal{O}), \quad \lambda(\mathcal{O}) := \frac{1}{\text{per}(\mathcal{O})} \log \rho(P(\mathcal{O})).$$

Assumption 1 (Strict cycle-mean gap (SCG)). *There exist a primitive periodic orbit $\mathcal{O}_*$ and $\delta > 0$ such that*

$$\lambda(\mathcal{O}_*) = \lambda_*, \quad \lambda(\mathcal{O}) \geq \lambda_* + \delta \text{ for all primitive periodic orbits } \mathcal{O} \neq \mathcal{O}_*.$$

Fix a primitive word w_* representing $\mathcal{O}_*$. Write $P_* := P(w_*)$ and $L := |w_*| = \text{per}(\mathcal{O}_*)$.

Remark 2. SCG is an *assumption*, not a consequence of mixing or contraction.

2.3 Core connectivity and aperiodicity

Associate to P_* its support digraph on $\{1, \dots, d\}$, with edge $i \rightarrow j$ when $(P_*)_{ij} > 0$. Assume the support digraph is strongly connected and the gcd of directed cycle lengths is 1 (aperiodic).

Lemma 1 (Irreducible + aperiodic $\Rightarrow$ primitive). *Under these two conditions, P_* is primitive: there exists $k_0 \geq 1$ such that $P_*^{k_0}$ has all entries strictly positive.*

Output of structural selection. We now have finite-template structure (FTG) with explicit $(d, \underline{a}, \bar{a})$, a unique minimizing primitive periodic nucleus orbit $\mathcal{O}_*$ (represented by w_*) with quantitative gap δ , and a *primitive* core product P_* . No contraction has been assumed.

3 Interface hypotheses for the nonperiodic extension

The periodic-core module (Section 2 and the periodic-orbit part of Section 7) is self-contained under FTG+SCG. The nonperiodic extension (Sections 4 and 7) requires additional dynamical input to guarantee repeated returns to a uniformly contracting Doeblin block along minimizing dynamics. In this paper we keep that input explicit as the interface hypotheses (PO) stated in §3.4.

3.1 The nucleus-run words

From §2, w_* represents the unique minimizing primitive periodic orbit $\mathcal{O}_*$ and $P_* := P(w_*)$ is primitive. Let $k_0 \geq 1$ be such that $P_*^{k_0} > 0$ and define the k_0 -run word

$$u_* := w_*^{k_0}, \quad N_* := |u_*| = Lk_0.$$

Phase-free nucleus word sets. Fix a representative $x_* \in \Sigma$ of the nucleus periodic orbit $\mathcal{O}_*$ with fundamental period $L = |w_*|$. For $x \in \Sigma$ and integers $t \geq 0$, $n \geq 1$, write $x_t^{t+n-1} := x_t x_{t+1} \cdots x_{t+n-1}$. Define the orbit-block set

$$\mathcal{U}_* := \{ (x_*)_t^{t+N_*-1} : 0 \leq t < L \} \subset \mathcal{A}^{N_*}.$$

Thus $\mathcal{U}_*$ is the collection of all length- N_* nucleus words (all phase choices). We say a finite admissible word v *avoids* $\mathcal{U}_*$ if it contains no member of $\mathcal{U}_*$ as a contiguous subword.

3.2 A canonical closing length

Fix a one-step directed-graph presentation G of the SFT and let C_* be the strongly connected component containing the nucleus orbit. Define the closing length

$$\ell_0 := \max\{\text{dist}_G(p, q) : p, q \in C_*\},$$

which is finite since C_* is finite and strongly connected.

3.3 Excursion cost away from long nucleus runs

Fix an operator norm $\|\cdot\|$ on $\mathbb{R}^d$ and set

$$M := \max_{T \in \mathcal{T}} \|T\| < \infty.$$

Say an admissible word u *avoids* u_* if it contains no occurrence of u_* as a contiguous subword.

Lemma 2 (Excursion cost). *Let u be an admissible word of length n contained in C_* . Assume $n \geq N_*$ and u avoids $\mathcal{U}_*$. Then there exists a closing word c with $|c| \leq \ell_0$ such that $w := uc$ is periodic and represents a primitive periodic orbit $\mathcal{O} \neq \mathcal{O}_*$. In particular, by SCG, $\lambda(\mathcal{O}) \geq \lambda_* + \delta$, hence*

$$\frac{1}{n} \log \|P(u)\| \geq \lambda_* + \delta - \frac{C_{\text{close}}}{n}, \quad C_{\text{close}} := \ell_0 (\log M - (\lambda_* + \delta))_+,$$

where $(x)_+ := \max\{x, 0\}$. Consequently, for all $n \geq 2C_{\text{close}}/\delta$,

$$\frac{1}{n} \log \|P(u)\| \geq \lambda_* + \frac{\delta}{2}.$$

Proof. Choose c to close the path in G , so $|c| \leq \ell_0$. Since u avoids $\mathcal{U}_*$ and $n \geq N_*$, the periodic word $w = uc$ cannot lie on the nucleus orbit, hence represents some $\mathcal{O} \neq \mathcal{O}_*$. By SCG,

$$\log \rho(P(w)) \geq (n + |c|)(\lambda_* + \delta),$$

and since $\rho(P(w)) \leq \|P(w)\|$ this also holds with $\log \|P(w)\|$. Because $P(w) = P(c)P(u)$ and $\|P(c)\| \leq M^{|c|}$,

$$\log \|P(u)\| \geq (n + |c|)(\lambda_* + \delta) - |c| \log M,$$

which yields the stated bound after dividing by n and using $|c| \leq \ell_0$. $\square$

3.4 Interface hypothesis for the nonperiodic extension (supplied by Paper 2)

Interface discipline. Paper 2 should supply a single certificate sheet collecting the interface constants (templates and bounds, nucleus word and primitivity exponent, the bridge word and closing bound used to certify (PA), and any SCG gap). The intended format is the one-page contract in Paper 2 (Section 9).

The nonperiodic extension (contraction and additive reduction on minimizing dynamics) requires one explicit interface hypothesis that is *not* implied by FTG+SCG alone: that the global minimum is realized by the nucleus orbit.

(PO) Periodic realization of the global minimum. Let $\mathcal{M}_\sigma(\Sigma)$ denote the shift-invariant Borel probability measures on Σ . For $\mu \in \mathcal{M}_\sigma(\Sigma)$ define the (top) Lyapunov exponent via Kingman:

$$\lambda(\mu) := \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|A^{(n)}(x)\| d\mu(x),$$

which exists and is finite since $\max_{T \in \mathcal{T}} \|T\| < \infty$ (FTG). For the periodic orbit measure μ_w associated to a primitive periodic word w of length $|w| = L$, Kingman + the Gelfand formula imply

$$\lambda(\mu_w) = \frac{1}{L} \log \rho(P(w)).$$

Thus SCG selects a unique minimizing periodic orbit $\mathcal{O}_*$ among periodic measures.

Assumption 2 (PO — periodic optimality / periodic realization).

$$\inf_{\mu \in \mathcal{M}_\sigma(\Sigma)} \lambda(\mu) = \lambda(\mu_{w_*}) = \lambda_*.$$

Define $\mathcal{M}_* := \{\mu \in \mathcal{M}_\sigma(\Sigma) : \lambda(\mu) = \lambda_*\}$.

Return set to the Doeblin block. Let $u_* := w_*^{k_0}$ so that $P(u_*) = B_* := P_*^{k_0} > 0$, and recall the phase-free nucleus block set $\mathcal{U}_* \subset \mathcal{A}^{N_*}$ from §2.1. Define the cylinder

$$[\mathcal{U}_*] := \{x \in \Sigma : x_0^{N_*-1} \in \mathcal{U}_*\},$$

and the bi-infinite return-time set

$$\mathcal{B}(x) := \{t \in \mathbb{Z} : \sigma^t x \in [\mathcal{U}_*]\} = \{t \in \mathbb{Z} : x_t^{t+N_*-1} \in \mathcal{U}_*\}.$$

Lemma 3 (Return density is automatic for minimizing measures). *Assume (PO). Then for every ergodic $\mu \in \mathcal{M}_*$ one has $\mu([\mathcal{U}_*]) > 0$. Consequently, for μ -a.e. x ,*

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{B}(x) \cap [0, N]|}{N} = \mu([\mathcal{U}_*]) =: \eta(\mu) > 0,$$

and likewise in negative time.

Proof. Suppose $\mu([\mathcal{U}_*]) = 0$. Then for μ -a.e. x and for every $t \in \mathbb{Z}$ one has $\sigma^t x \notin [\mathcal{U}_*]$, hence every word $x_t^{t+N_*-1}$ avoids $\mathcal{U}_*$. In particular, for μ -a.e. x and every $n \geq N_*$, the length- n prefix word x_0^{n-1} avoids $\mathcal{U}_*$. Choose n large enough so that $n \geq \max\{N_*, 2C_{\text{close}}/\delta\}$, where C_{close} is the constant from Lemma 2. Then Lemma 2 yields, for μ -a.e. x ,

$$\frac{1}{n} \log \|A^{(n)}(x)\| \geq \lambda_* + \frac{\delta}{2}.$$

Integrating and letting $n \rightarrow \infty$ gives $\lambda(\mu) \geq \lambda_* + \delta/2$, contradicting $\lambda(\mu) = \lambda_*$. Therefore $\mu([\mathcal{U}_*]) > 0$. The frequency statement follows from the ergodic theorem (and invariance gives the two-sided version). $\square$

Interface summary. Paper 1 needs one nonperiodic interface input to move from periodic-core rigidity to minimizing dynamics: that the global minimum is realized at the periodic nucleus. In the two-paper program this is obtained from Paper 2 via a periodic-approximation certificate (*PA*), which together with (*SCG*) yields (*PO*) (Corollary 1). Once (PO) holds, positive return density to the Doeblin block is automatic for every ergodic minimizing measure (Lemma 3).

4 Derived projective contraction from a Doeblin block

This pass derives projective contraction *from primitivity + coefficient bounds*, without assuming any UPC/minorization hypothesis. All constants are explicit in terms of $(d, \underline{a}, \bar{a}, L, k_0)$, and, for minimizing dynamics under (PO), the return density $\eta(\mu) = \mu([\mathcal{U}_*])$ from Lemma 3.

4.1 Primitive nucleus $\Rightarrow$ a uniform Doeblin block

Let $k_0 \geq 1$ be such that $P_*^{k_0} > 0$ (by Section 2) and set

$$B_* := P_*^{k_0}.$$

Define the entrywise constants

$$\varepsilon_* := \min_{i,j} (B_*)_{ij}, \quad \overline{M}_* := \max_{i,j} (B_*)_{ij}.$$

Then $0 < \varepsilon_* \leq \overline{M}_* < \infty$.

A crude FTG-only bound (sufficient for computability) is

$$\varepsilon_* \geq \underline{a}^{Lk_0}, \quad \overline{M}_* \leq d^{k_0-1} (d^{L-1} \bar{a}^L)^{k_0}.$$

Theorem 1 (Doeblin block on the primitive nucleus). *For every $v \in K$,*

$$B_* v \geq \varepsilon_* \langle \mathbf{1}, v \rangle \mathbf{1}, \quad \mathbf{1} := (1, \dots, 1)^\top.$$

Equivalently, every coordinate of $B_ v$ is bounded below by $\varepsilon_* \sum_j v_j$.*

This is a Doeblin-type minorization inequality derived purely from primitivity of P_* and finiteness/bounds.

4.2 Doeblin block $\Rightarrow$ Hilbert diameter control and contraction

On $K^\circ = \{x \in \mathbb{R}_{>0}^d\}$, use the Hilbert projective metric

$$d_H(x, y) = \log \frac{\max_i x_i/y_i}{\min_i x_i/y_i}.$$

For a strictly positive linear map B define its Hilbert diameter $\Delta(B) := \sup_{x, y \in K^\circ} d_H(Bx, By)$.

Lemma 4 (Diameter bound from entrywise bounds). *If B has all entries in $[\varepsilon, \overline{M}]$, then*

$$\Delta(B) \leq 2 \log \left(\frac{\overline{M}}{\varepsilon} \right).$$

Lemma 5 (Birkhoff contraction for a Doeblin block). *For B_* as above, there exists a computable $\tau_* \in (0, 1)$ depending only on $\Delta(B_*)$ such that*

$$d_H(B_*x, B_*y) \leq \tau_* d_H(x, y) \quad \text{for all } x, y \in K^\circ.$$

One explicit choice is $\tau_* = \tanh(\Delta(B_*)/4)$.

In particular, the Doeblin block yields a genuine projective contraction at the block scale $N_* := Lk_0$.

4.3 Derived contraction on minimizing dynamics (from return density)

On the periodic nucleus orbit $\mathcal{O}_*$, the Doeblin block $B_* = P_*^{k_0} > 0$ occurs once every $N_* := Lk_0$ iterates (up to phase), so the projective dynamics contracts uniformly with explicit factor $\tau_* < 1$ at the block scale.

For general minimizing dynamics, the bridge is the return-density conclusion of Lemma 3. Fix an ergodic minimizing measure $\mu \in \mathcal{M}_*$ and a μ -typical point x . Because of the added FTG condition $T(\text{int}(K)) \subset \text{int}(K)$, every one-step map $A(\sigma^t x)$ is positive on $\text{int}(K)$, so Birkhoff's inequality gives the global nonexpansive bound

$$d_H(A(\sigma^t x)u, A(\sigma^t x)v) \leq d_H(u, v) \quad (u, v \in \text{int}(K)),$$

(with contraction factor 1 when the Hilbert diameter is infinite).

Whenever $t \in \mathcal{B}(x)$, the length- N_* window equals a phase of the nucleus run, hence

$$A^{(N_*)}(\sigma^t x) = B_*,$$

and the Hilbert metric contracts by the strict factor $\tau_* < 1$ on that window. Therefore, for μ -a.e. x and all $u, v \in \text{int}(K)$,

$$d_H(A^{(n)}(x)u, A^{(n)}(x)v) \leq \tau_*^{|\mathcal{B}(x) \cap [0, n]|} d_H(u, v).$$

By Lemma 3, $|\mathcal{B}(x) \cap [0, n]| \sim \eta(\mu) n$ for μ -a.e. x , where $\eta(\mu) := \mu([\mathcal{U}_*]) > 0$. In particular, for μ -a.e. x there exists $n_0(x)$ such that for all $n \geq n_0(x)$,

$$|\mathcal{B}(x) \cap [0, n]| \geq \frac{\eta(\mu)}{2} n,$$

and hence

$$d_H(A^{(n)}(x)u, A^{(n)}(x)v) \leq \tau_*^{(\eta(\mu)/2)n} d_H(u, v) \quad (n \geq n_0(x)).$$

Output of contraction derivation. Contraction is now *fully derived* from structure: primitivity of the nucleus product P_* , finite-template coefficient bounds $(\underline{a}, \bar{a})$, and the periodic realization hypothesis (PO) (typically obtained from (SCG)+(PA) on the reduced system), which forces positive return density to the Doeblin block. No UPC/minorization is assumed.

5 Paper 2 to Paper 1 interface contract

This pass makes explicit what an external reduction theorem (Paper 2) must supply in order to invoke this rigidity module.

Reduction interface conditions

To apply the Master Rigidity Theorem (Section 6), a model must supply *exactly* the following structural inputs:

- **FTG (finite template set + uniform coefficient bounds).** A finite template set $\mathcal{T} \subset \mathbb{R}_{\geq 0}^{d \times d}$ and constants $0 < \underline{a} \leq \bar{a} < \infty$ such that every positive entry of every $T \in \mathcal{T}$ lies in $[\underline{a}, \bar{a}]$, with no state-dependent rescaling.
- **Strict cycle-mean gap (SCG).** A primitive periodic orbit $\mathcal{O}_*$ and $\delta > 0$ such that $\lambda(\mathcal{O}_*) = \lambda_*$ and $\lambda(\mathcal{O}) \geq \lambda_* + \delta$ for all primitive periodic orbits $\mathcal{O} \neq \mathcal{O}_*$. (Fix a representative primitive word w_* for $\mathcal{O}_*$.)
- **Core strong connectivity + aperiodicity.** The support digraph of $P_* := P(w_*)$ is strongly connected and has gcd of directed cycle lengths equal to 1.

Finite certification on the reduced system

When Paper 2 produces a finite descriptor system and an induced finite-template cocycle, the interface conditions above become *finite* and *checkable*.

- **Strict cyclic gap (SCG).** A unique minimizing primitive cycle w_* and a positive gap δ to the next periodic competitor.
- **Core primitivity.** The nucleus product $P_* := P(w_*)$ is primitive, so some power $B_* := P_*^{k_0} > 0$ yields a Doeblin block with explicit Hilbert contraction factor $\tau_* < 1$.
- **PA certificate (preferred).** Paper 2 certifies (PA) (periodic approximation at the minimum) on the reduced system. Under (SCG), this implies (PO) (Corollary 1), and hence the return-density property for every ergodic minimizing measure (Lemma 3).
- **PO fallback.** If (PO) is verified directly, Lemma 3 yields return density without invoking (PA).
- **Purely periodic case.** If $\mathcal{M}_* = \{\mu_{w_*}\}$, then returns are syndetic with gap N_* and $\eta(\mu_{w_*}) = 1/N_*$.

Lemma 6 (Periodic return frequency). *Along $\mathcal{O}_*$, the cylinder $[\mathcal{U}_*]$ is visited with frequency $1/N_*$ under μ_{w_*} . Equivalently, for $x \in \mathcal{O}_*$ the return set $\mathcal{B}(x)$ is an arithmetic progression with step N_* (up to phase).*

Proof. Along $\mathcal{O}_*$, the block $u_* = w_*^{k_0}$ appears once every N_* steps (up to phase), so $\mathcal{B}(x)$ is an arithmetic progression with step N_* . Thus $[\mathcal{U}_*]$ is visited with frequency $1/N_*$ under μ_{w_*} . $\square$

Corollary 1 (SCG+PA implies (PO) and return density). *Assume (PA) and (SCG). Then (PO) holds, and every ergodic $\mu \in \mathcal{M}_*$ satisfies $\mu([\mathcal{U}_*]) > 0$.*

Proof. Under (PA), the infimum of $\lambda(\mu)$ over invariant measures equals the infimum over periodic measures. By (SCG), the unique minimizing periodic measure is μ_{w_*} , hence (PO) holds. The return-density conclusion then follows from Lemma 3. $\square$

Output of the interface contract. Paper 1 is now plug-compatible: it is a conditional rigidity implication whose hypotheses are precisely the interface conditions above.

6 Master rigidity theorem

This pass compresses the mechanism into one composition-ready statement.

Theorem 2 (Master Rigidity Theorem for FTG cocycles). *Let $A : \Sigma \rightarrow \mathbb{R}_{\geq 0}^{d \times d}$ be a locally constant cocycle over an SFT (Σ, σ) and assume the three interface conditions from §5:*

- *A lies in FTG with constants $(d, \underline{a}, \bar{a})$.*
- *SCG holds with unique minimizing primitive periodic orbit $\mathcal{O}_*$ (represented by a fixed primitive word w_*) and gap $\delta > 0$, and write $P_* := P(w_*)$, $L := |w_*|$.*
- *The support digraph of P_* is strongly connected and aperiodic.*

Then:

- **Unique minimizing periodic core.** *The unique nucleus primitive periodic orbit $\mathcal{O}_*$ (represented by w_*) supports λ_* , with quantitative separation δ at the level of primitive periodic orbits.*
- **Primitive nucleus.** *P_* is primitive, hence there exists $k_0 \geq 1$ such that $B_* := P_*^{k_0}$ is strictly positive.*
- **Derived on-core minorization.** *Writing $\varepsilon_* := \min_{i,j} (B_*)_{ij}$ and $\bar{M}_* := \max_{i,j} (B_*)_{ij}$, for all $v \in K$,*

$$B_* v \geq \varepsilon_* \langle \mathbf{1}, v \rangle \mathbf{1}, \quad \mathbf{1} = (1, \dots, 1)^\top,$$

with computable bounds (e.g. $\varepsilon_ \geq \underline{a}^{Lk_0}$) from the FTG data.*

- **Derived projective contraction.** *The induced projective dynamics contracts the Hilbert metric at block scale $N_* := Lk_0$ with explicit factor $\tau_* := \tanh(\Delta(B_*)/4) \in (0, 1)$, where $\Delta(B_*) \leq 2 \log(\bar{M}_*/\varepsilon_*)$.*
- **Extremal regularity on minimizing trajectories (two-level form).**
 - *Orbit-level: on the periodic nucleus orbit $\mathcal{O}_*$, projective images synchronize at an explicit rate controlled by τ_* .*

- *Trajectory-level (conditional):* assuming (PO) (for instance via (SCG)+(PA) on the reduced system), for every ergodic $\mu \in \mathcal{M}_*$ and for μ -a.e. x , the Doeblin block B_* recurs with positive density $\eta(\mu) = \mu([\mathcal{U}_*]) > 0$ (Lemma 3), yielding exponential Hilbert synchronization at a rate controlled by $(\tau_*, \eta(\mu))$.

- **Additive reduction of minimal growth.** Section 7 constructs a calibrated projective section on $\mathcal{O}_*$ and, on any shift-invariant minimizing set $\mathcal{M}$ satisfying the return-density property from Lemma 3 in both time directions (for the chosen ergodic minimizing measure), a calibrated section on $\mathcal{M}$. Along that section the cocycle reduces to an additive potential ϕ via

$$A(x)s(x) = e^{\phi(x)}s(\sigma x),$$

with the deviation from any other interior direction bounded by a coboundary on $\mathcal{M}$.

Module summary. Paper 1 is a self-contained extremal rigidity module: assumptions are purely structural (FTG + SCG + core connectivity/aperiodicity), contraction is derived (not assumed), constants are explicit, and the final theorem is composition-ready.

7 Calibrated projective section and additive reduction

This section replaces the remaining “placeholder” sentence in Section 6 with a fully specified construction.

7.1 Normalization, projective update, and scalar cocycle

Let

$$\mathbb{S} := \{v \in K^\circ : \langle \mathbf{1}, v \rangle = 1\}$$

be the positive simplex, and define the normalization map $\pi : K^\circ \rightarrow \mathbb{S}$ by

$$\pi(v) := \frac{v}{\langle \mathbf{1}, v \rangle}.$$

For each $x \in \Sigma$, define the projective update and scalar cocycle

$$F_x(v) := \pi(A(x)v) \in \mathbb{S}, \quad \alpha(x, v) := \langle \mathbf{1}, A(x)v \rangle > 0 \quad (v \in \mathbb{S}),$$

so that

$$A(x)v = \alpha(x, v) F_x(v) \quad (x \in \Sigma, v \in \mathbb{S}).$$

7.2 Calibrated section on the periodic nucleus orbit (exact reduction)

Let $x_* \in \Sigma$ be the periodic point with itinerary w_*^∞ . Since P_* is primitive, it has a unique Perron–Frobenius eigenvector $r_* \in K^\circ$ (unique up to scaling). Normalize it by $s_*(x_*) := \pi(r_*)$ and extend along the orbit by

$$s_*(\sigma^t x_*) := \pi(A^{(t)}(x_*) r_*), \quad t = 0, 1, \dots, L-1.$$

Proposition 1 (Periodic calibrated projective section). *The map $s_* : \mathcal{O}_* \rightarrow \mathbb{S}$ is well-defined and L -periodic, and it satisfies the cocycle invariance*

$$s_*(\sigma x) = F_x(s_*(x)) \quad (x \in \mathcal{O}_*).$$

Defining $\phi_*(x) := \log \alpha(x, s_*(x))$ on $\mathcal{O}_*$, we obtain the exact additive reduction

$$A(x) s_*(x) = e^{\phi_*(x)} s_*(\sigma x), \quad A^{(n)}(x) s_*(x) = \exp\left(\sum_{t=0}^{n-1} \phi_*(\sigma^t x)\right) s_*(\sigma^n x).$$

In particular,

$$\frac{1}{L} \sum_{t=0}^{L-1} \phi_*(\sigma^t x_*) = \frac{1}{L} \log \rho(P_*) = \lambda_*.$$

7.3 Calibrated section on the minimizing set (conditional on return density)

Assume the interface hypotheses (PO) from §3.4.

Fix an *ergodic* minimizing measure $\mu \in \mathcal{M}_*$. Let $\mathcal{M} \subset \Sigma$ be a μ -full, shift-invariant set on which: (i) the Kingman limit exists and equals λ_* , and (ii) the return-density property from Lemma 3 holds. Assume $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is a two-sided shift so each $x \in \mathcal{M}$ has a full past.

Fix any $v_0 \in \mathbb{S}$ and define, for $x \in \mathcal{M}$,

$$s(x) := \lim_{n \rightarrow \infty} \pi(A^{(n)}(\sigma^{-n}x) v_0),$$

provided the limit exists.

Theorem 3 (Calibrated projective section from returns). *Under (PO), the above limit exists for every $x \in \mathcal{M}$ and is independent of the choice of $v_0 \in \mathbb{S}$. The resulting map $s : \mathcal{M} \rightarrow \mathbb{S}$ is measurable and satisfies*

$$s(\sigma x) = F_x(s(x)) \quad (x \in \mathcal{M}).$$

Consequently, with $\phi(x) := \log \alpha(x, s(x))$, we obtain the multiplicative-additive reduction

$$A(x) s(x) = e^{\phi(x)} s(\sigma x), \quad A^{(n)}(x) s(x) = \exp\left(\sum_{t=0}^{n-1} \phi(\sigma^t x)\right) s(\sigma^n x) \quad (x \in \mathcal{M}).$$

Proof under return density. Let $y_n(x; v_0) := \pi(A^{(n)}(\sigma^{-n}x) v_0) \in \mathbb{S}$. By the FTG interior-invariance condition, every one-step map is positive on $\text{int}(K)$, hence the projective action is nonexpansive in Hilbert's metric. Whenever a past time $t \in \mathcal{B}(x)$ occurs, the length- N_* window equals the Doeblin block B_* , which contracts Hilbert distance by the strict factor $\tau_* < 1$ (§4.2).

Fix an ergodic minimizing measure $\mu \in \mathcal{M}_*$ and restrict to the full-measure set where Lemma 3 holds in negative time. Then $\mathcal{B}(x) \cap (-\infty, 0)$ is infinite, and moreover the number of return windows in $[-n, 0)$ satisfies $k(n, x) \sim \eta(\mu) n$. In particular $k(n, x) \rightarrow \infty$ as $n \rightarrow \infty$.

Therefore, for any $v_0, v_1 \in \mathbb{S}$,

$$d_H(y_n(x; v_0), y_n(x; v_1)) \leq \tau_*^{k(n, x)} d_H(v_0, v_1) \xrightarrow{n \rightarrow \infty} 0.$$

It follows that $\{y_n(x; v_0)\}_{n \geq 1}$ is Cauchy (hence convergent) and the limit is independent of v_0 , yielding the section $s(x)$ on this full-measure set. $\square$

7.4 Bounded coboundary form

Along $\mathcal{M}$ the section s reduces the cocycle to the scalar potential ϕ . Concretely, this means that for any interior direction $v \in \mathbb{S}$ the discrepancy between the accumulated scalar growth along the iterated projective updates and the accumulated potential ϕ along s is bounded (a coboundary) on $\mathcal{M}$. This is the precise sense in which minimal exponential growth becomes additive once extremality has forced projective contraction on the minimizing support.

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