

Syncré Form Theory (SFT): Coverage Phase Fields, Witness Engines, Coupled Targets, and Stability Signatures

Drew Higgins

Abstract

Syncré Form Theory (SFT) provides a target-correct, auditable semantics for phase-coverage statements on slices or declared subsystems. A declared context specifies a slice/subsystem, a circle- or torus-valued phase observable, a target rule (including coupling conventions), a witness mechanism, and a perturbation model. Within the Cyclic Observability Class (COC), the SFT engine resolves a declared claim into exactly one checkable object: a Witness Stability Certificate (WSC) proving coverage on the correct target with explicit robustness margins, or a Witness Obstruction (WOB) naming the first failed validator predicate in a finite codebook with evidence pointers. The theory packages standard forcing mechanisms (equivariant circle actions, loop degree, holonomy) together with a coupled-target calculus for multi-cycle observables and a conservative target-inference theorem producing coupled subtorus descriptions or obstructions at a declared tolerance. The release bundle includes schemas, audits, benchmarks, a witness atlas tied to shipped artifacts, and hash-verified reproducibility scripts.

Reader guide, notation, and what is proved

Core objects

A claim in SFT is stated on a *declared context*:

- a sliced arena $M = \bigsqcup_{s \in \mathbb{R}} \Sigma_s$ and a declared slice or subsystem on which phases are observed,
- a phase observable into a circle $\mathbb{T}_P = \mathbb{R}/P\mathbb{Z}$ or a torus $\mathbb{T}_{\mathbf{P}}$,
- a target rule that selects the correct target set (circle, full torus, or a coupled subtorus coset CST_s) together with the baseline convention (Haar on the target, or coupled Haar on CST_s),
- a witness mechanism (action, loop degree, holonomy, or coupled-target mechanism) and a perturbation model that fixes stability meaning.

Certificate-or-obstruction meaning

Within the Cyclic Observability Class (COC), every declared claim resolves to exactly one audited object:

- **WSC (Witness Stability Certificate):** coverage on the correct target with explicit robustness margins,
- **WOB (Witness Obstruction):** a canonical obstruction code (first failed validator predicate) with evidence pointers.

Acronyms

Token	Meaning
SCL	Syncré Coverage Law: slice-wise surjectivity of a phase field on the declared target
COC	Cyclic Observability Class: the declared regime where predicates and witness engines apply
WSC / WOB	Witness Stability Certificate / Witness Obstruction (certificate-or-obstruction output types)
CST / CST _s	Coupled subtorus coset (possibly slice-dependent), the correct target under coupling
AWT	Action witness template (equivariant circle action forcing)
LDW	Loop degree witness (nonzero degree on an embedded loop)
HW	Holonomy witness (nontrivial holonomy multiple on a loop)
CTP / CHT	Correct-target principle / coupled Haar target baseline convention
LCM / HMR	Loop-closure margin / holonomy-margin rule (robustness margins stored in WSC)
RNF / NDB / MXC	Nonresonant increment hypothesis / noise-to-diffusion bridge / mixing certificate inputs

What is proved

The technical content combines standard forcing mechanisms with a new verifiable decision contract.

- Forcing theorems for coverage from equivariant circle actions (AWT), loop degree (LDW), and holonomy (HW).
- Target-correctness axioms and a coupled-target calculus: under coupling, coverage and baselines are asserted on CST_s, not on the ambient product torus.
- Predicate-exhaustion exhaustiveness: inside COC, every declared claim admits either a WSC or a WOB, with WOB codes determined by first predicate failure.
- Conservative target-inference theorem: from data and a declared tolerance, either produce a stable CST description (with margins) or return a finite obstruction (ambiguity, insufficient excitation, resonance masking).

1 Introduction and contributions

Phase observables occur whenever a system carries a clock-like quantity: angle, time-of-day, wave phase, oscillator phase, or the phase of a periodic drive. At a fixed global instant, such a phase can vary across space or across a declared subsystem, producing a *phase field*. SFT formalizes a common slice-wise question:

On a declared slice or subsystem, does the phase field attain every phase value on the correct target, and is this statement stable under a declared perturbation model?

Two difficulties force discipline.

- **Forcing must be explicit.** Coverage is not inferred from intuition. SFT requires a declared witness mechanism (action, degree, holonomy, or coupled-target forcing) that can be audited.
- **Targets must be correct under coupling.** In multi-cycle settings, coupling can restrict the image to a coupled subtorus coset CST_s . Coverage and any baseline distribution must then be asserted on CST_s , not on the ambient product torus.

SFT packages these requirements as a verifiable decision contract. Given a declared context, the engine produces exactly one audited object type inside COC:

- a **WSC** proving coverage on the correct target with explicit stability margins, or
- a **WOB** naming the first failed validator predicate in a finite codebook, with evidence pointers.

What is new in this work

The forcing mechanisms themselves rely on standard pillars (degree theory, compact group actions, torus dynamics, diffusion on compact manifolds). The contribution is the way these pillars are assembled into a target-correct, auditable certificate system with a sharp boundary.

- **Target-correct certificate semantics under coupling.** Coupling is treated as part of meaning: certificates and baselines are stated on CST_s when constraints force the image into a coupled coset.
- **Predicate-exhaustion exhaustiveness.** Inside COC, the validator predicate suite makes outcomes exhaustive: a declared claim admits either a WSC or a WOB, with canonical first-failure codes.
- **Quantitative stability margins as first-class objects.** WSC artifacts store margins (LCM/HMR and derived modulus fields) that make stability claims checkable and portable.
- **Conservative target inference with obstruction fallback.** A data-to-certificate workflow either outputs a stable coupled-target description with margins or returns a finite obstruction (ambiguity, insufficient excitation, resonance masking).
- **Artifact-backed reproducibility.** The release bundle contains schemas, audits, benchmarks, and hash-verified build scripts linking theorems to checkable objects.

Higher target and proxy

The long-range target is a relational claim about form:

Relational Form Principle. Form is an invariant of embeddedness. A point (or any reference marker) does not carry its form as a standalone property; its form is determined by, and inseparable from, how it is embedded in the whole structure.

SFT begins with a proxy that preserves the same invariance logic while remaining directly observable:

Proxy Identity Principle. The Syncré layer is a minimal observable proxy in which forcing, finite obstructions, stability margins, and target correctness are theorem-driven and recorded as checkable objects.

In this proxy layer, a single global instant is expressed as a phase field across space (or across an embedded cyclic subsystem). Local values are meaningful only relative to the whole slice or subsystem, and correctness is enforced by certificates on the correct target (or by finite obstructions when certification is impossible).

Venue positioning note

The “RFP / PIP” layer is philosophically interesting but academically expensive

The “higher relational claim” framing can be polarizing in a math venue because it reads like a manifesto. It may fit better as:

an intro essay section, clearly marked as motivation, or
a separate companion note.

If you want acceptance in technical venues, the safest move is:

keep it, but quarantine it so it cannot be confused with theorems.

What this paper proves

- A precise coverage law for cycle-valued phase fields on sliced arenas, including the correct lift/holonomy interfaces needed for nonlinear deformation.
- Forcing results: under minimal cyclic symmetry plus nondegenerate equivariance (or nonzero winding/holonomy on an embedded loop), full-cycle coverage is structurally forced on the appropriate target.
- A finite obstruction taxonomy: when a coverage claim fails in the declared regime, at least one named structural obstruction must occur, and the correct response is an obstruction report with evidence.
- Robustness: witnesses persist under explicit quantitative margins, and quantitative anti-collapse bounds follow under stated regularity hypotheses.
- Correct target selection under coupling: when constraints restrict the image, coverage and uniformity must be stated on the coupled target (CST) with the coupled Haar baseline.
- Convergence upgrades when justified: under nonresonant drives, time averages converge to Haar on the correct target [4, 6]; under explicit diffusion/noise models, uniformity improves at certified rates [8, 10].

Universality as an exhaustiveness statement inside COC

In SFT, a universality statement is a statement about *resolution* within a declared regime.

- There is a named applicability class of declared contexts, the Cyclic Observability Class (COC).
- On COC, phase-coverage claims reduce to a finite set of witness mechanisms (action, degree, holonomy, coupled targets) together with a finite validator predicate suite.
- Within COC, any declared claim has an audited outcome type: either a Witness Stability Certificate (WSC) or a Witness Obstruction (WOB) with a canonical code.

COC is broad enough to cover many cyclic observability settings (rotation and periodic forcing, embedded loops and defects, multi-cycle forcing with coupling), while still keeping a sharp boundary: outside COC the engine returns boundary obstructions rather than extending hypotheses by guesswork.

Scope Wall (what is claimed and what is not)

This section is designed to be hard to misread.

What is claimed

- SFT defines a maximal named class, the *Cyclic Observability Class (COC)*, for which Syncré statements are meaningful.
- For every context declared in COC, the engine produces either:
 - a **WSC** (Witness Stability Certificate) with explicit margins and a validator predicate ledger, or
 - a **WOB** (Witness Obstruction) with a canonical code and evidence.
- Target correctness is part of meaning: when coupling restricts the image to a coupled subtorus CST_s , statements are made *on* CST_s with coupled Haar baseline.

What is not claimed

- Not every dynamical system is in COC.
- Deterministic transport is not asserted to be universally mixing.
- Empirical checks are shipped as evidence objects; they are not presented as proofs.

What exists as runnable artifacts

- Schemas and contracts: `engine/schemas/` and `engine/contracts/`
- Audit scripts: `engine/audit/`
- Example WSC/WOB/CC objects: `engine/artifacts/`
- Benchmarks producing evidence payloads: `engine/benchmarks/`

Well-posedness spine: what a universality claim means in SFT

SFT treats universality as a statement with an explicit boundary. The boundary is not a philosophical warning; it is part of the formal interface. Every claim is posed in a *declared context* and every answer is a finite object. This is the only sense in which SFT uses the word *meaning*: a claim is meaningful exactly when it has an auditable certificate-or-obstruction output.

Definition 1 (Declared SFT claim). *A declared SFT claim is a tuple*

$$\mathbf{q} = (\mathbf{c}, \mathbf{T}, \mathbf{S}),$$

where $\mathbf{c}$ is a declared context (with a phase observable and an admissible witness mechanism), $\mathbf{T}$ is the declared correct target (circle/torus/coupled target) together with its baseline convention (Haar on the target; coupled Haar on CST when coupling is present), and $\mathbf{S}$ is the concrete Synchré statement being tested (coverage, stability, and any declared upgrade).

Well-posedness axiom (WSC/WOB meaning). For every declared claim $\mathbf{q}$, the SFT engine returns exactly one of the following two outputs.

- **WSC (Witness Stability Certificate).** A finite certificate proving the stated claim on the declared correct target, together with explicit margins that make the claim stable in the declared perturbation model.
- **WOB (Witness Obstruction).** A finite obstruction report from a named catalog, with evidence, asserting that certification is blocked or that the claim is inapplicable in the declared regime.

The claim is *meaningful in SFT* if and only if it is posed as such an engine query and therefore admits a WSC/WOB output.

The sharp boundary can be read in one glance.

Where the claim lives	Correct output
Inside COC (phase observable + witness route + target rule)	WSC or a finite WOB (structural failure mode)
Outside COC (missing phase / missing witness route / wrong target rule)	Boundary-type WOB (inapplicable or ill-posed)

How to read any “universality” sentence in this project.

- First check whether the context is declared as belonging to COC (the checklist is part of the definition).
- Then check the target: circle, full torus, or a coupled target CST.
- Then check the object-level output: a WSC (with margins) or a WOB (with a named obstruction code and evidence).

This spine is the project’s scope wall in mathematical form. It prevents universality from becoming vague: every use of the word is anchored to the named class (COC) defined by the interface conditions and to an explicit certificate-or-obstruction output.

Target correctness axiom block (CTP/CHT). SFT treats *target correctness* as part of the meaning of any certificate-or-obstruction claim. A declared claim must state (i) the *target type* (circle / full torus / coupled target) and (ii) the *baseline measure* used for any comparison, stability, or convergence upgrade.

- **Circle or full torus targets.** When the correct target is a compact group target $\mathbb{T}_P$ or $\mathbb{T}_{\mathbf{P}}$, the baseline is the (unique) Haar probability measure on that target.
- **Coupled targets (CST).** When coupling constraints restrict the image to a coupled subtorus coset,

$$\text{CST}_s := \{y \in \mathbb{T}_{\mathbf{P}} : Ay = \Psi(s)\}, \quad A \in \mathbb{Z}^{r \times d},$$

then the correct target is CST_s (not the ambient torus) and the baseline is the (unique) translation-invariant probability measure on CST_s . Equivalently, for any parameterization $\eta_s : \mathbb{T}_{P_*}^m \rightarrow \text{CST}_s$, the baseline is the pushforward of Haar:

$$\mu_{\text{H},s}^{\text{CST}} := (\eta_s)_\# \mu_{\text{H}}^{(m)}.$$

CTP (Coupled Target Principle). If the declared context includes coupling constraints yielding a coupled target CST_s , then all coverage statements, stability margins, and upgrades are to be interpreted *relative to* CST_s . Using the ambient torus as the target in this regime is not a weaker claim; it is a *different* claim.

CHT (Coupled Haar Theorem, as baseline convention). In the coupled regime, the baseline for comparisons is $\mu_{\text{H},s}^{\text{CST}}$. Any convergence or mixing claim must specify this baseline when the target is CST_s .

Corollary (wrong target/baseline forces a boundary obstruction). If a declared claim uses a target/baseline pair incompatible with the declared coupling structure (e.g., it uses full-torus Haar when the correct target is CST_s), then the claim is *ill-posed in SFT*. The correct engine output in that case is the coupling-mismatch obstruction `WOB-06-coupling-mismatch`.

Target correctness decision flow

This page is the canonical interface for selecting the *correct target* and *baseline* in SFT. It is written to match the WSC schema fields `target_type` and `baseline`. The point is simple: if the target is chosen correctly, the engine can attempt certification; if the target is chosen incorrectly, the correct output is a named obstruction.

Decision	Action (what you declare)	If violated, output
A. Is the claim in COC?	Declare a phase observable and an admissible witness route (rotation / loop-degree / holonomy / multi-cycle) and state the target rule you are using.	If any ingredient is missing, the claim is outside COC: boundary-type obstruction (typically WOB-01).
B. Is coupling present?	If the model includes coupling constraints across phases (an integer constraint matrix χ or an equivalent coupling law), set <code>target_type=CST</code> and declare <code>baseline=coupledHaaronCS</code> T.	Using circle/torus baseline under coupling, or failing to compute/declare CST: WOB-06 .
C. Single-cycle circle observable?	If the witness route is a circle action or a single loop/defect witness producing an integer $k \in \mathbb{Z}$, set <code>target_type=circle</code> (or <code>embedded_circle</code> for subsystem-only claims) and <code>baseline=Haar</code> on $\mathbb{T}_P$.	If every admissible witness integer is 0: WOB-02 . If no loop/orbit exists in the declared domain: WOB-01 .
D. Multi-cycle torus observable?	If the witness route is a $\mathbb{T}^k$ action with response map induced by an integer matrix χ , compute the effective rank $r := \text{rank}(\chi)$. • If $r = k$, set <code>target_type=torus</code> and <code>baseline=Haar</code> on $\mathbb{T}^k$. • If $r < k$, set <code>target_type=subtorus_rank_r</code> and <code>baseline=Haar</code> on the induced r-torus (the χ-image).	Claiming full-torus forcing when the response rank is deficient (or rational locking collapses the sweep): WOB-10 .
E. Target-aware upgrades?	Any convergence or mixing upgrade must explicitly name the baseline it converges to. If the upgrade requires diffusion/mixing assumptions, record them and point to the declared evidence artifact.	Mixing/convergence claimed without an admissible hypothesis package: WOB-07 .

Allowed upgrades (summary).

Target	Baseline	Upgrades allowed (when justified)
Circle / embedded circle	Haar on $\mathbb{T}_P$	Stability margins (LCM/HMR), anti-collapse bounds, and convergence-to-Haar if a mixing model is declared.
Full torus $\mathbb{T}^k$	Haar on $\mathbb{T}^k$	Stability margins; rank-aware coverage; convergence-to-Haar if a mixing model is declared.
Subtorus rank r	Haar on induced r -torus	Same upgrades, but always relative to the induced target (never the ambient torus unless $r = k$).
CST (coupled target)	Coupled Haar on CST	All upgrades, but always relative to coupled Haar; any non-coupled baseline is ill-posed (CTP/CHT).

Primitives and the engine theorem

SFT is organized around five primitives.

- **Arena and slicing.** A system is modeled as an arena M equipped with slices Σ_s representing global instants.
- **Phase field.** A cycle-valued observable $\Phi : M \rightarrow \mathbb{T}_P$ assigns phase labels to points, with $\mathbb{T}_P$ a circle or torus of periods.
- **Frame equivalence (FEQ).** Descriptions that preserve slice correspondence and phase meaning are treated as equivalent so conclusions are invariant under admissible frame changes.
- **Witness mechanism.** Coverage is asserted only via a declared forcing route (action, loop degree, holonomy, or coupled-target forcing) in the declared model class.
- **Target correctness.** Under coupling, the correct target is a coupled subtorus coset CST_s with coupled Haar baseline; otherwise the target is the full torus with Haar baseline.

Engine theorem (interface soundness and exhaustiveness inside COC)

Fix a declared context in the Cyclic Observability Class (COC), including a target rule, witness mechanism, and perturbation model. Then the engine output type is exhaustive:

- **WSC output.** A witness stability certificate (WSC) on the correct target, recording the witness integer data and explicit margins.
- **WOB output.** A witness obstruction (WOB) whose code is the first failed validator predicate in the canonical predicate order, with evidence pointers meeting the code checklist.

The WSC predicates imply the stated coverage claim in the declared regime (soundness), and predicate exhaustion implies that a declared claim cannot fail without producing an explicit WOB code inside COC (exhaustiveness).

Certification dichotomy (declared regime)

Declared regime

Fix a regime consisting of:

- a sliced arena and a declared observation domain (global slice, embedded loop, or coupled target),
- a circle- or torus-valued phase observable in $\mathbb{T}_{\mathbf{P}}$,
- either (i) a genuine $S^1/\mathbb{T}^k$ symmetry action with an equivariance relation, or (ii) an embedded loop/defect subsystem supporting degree/holonomy,
- a declared perturbation model with explicit margins (LCM/HMR),
- in coupled settings, a declared target-correctness rule selecting CST_s and coupled Haar baseline.

Theorem (soundness and completeness of the interface)

For any stated claim target within the declared regime, the engine returns exactly one of the following outputs, and each output is independently checkable:

- **Certificate.** A witness stability certificate (WSC) on the correct target, recording a witness integer (degree/holonomy/free-coordinate degree), explicit stability margins, and the correct statistical baseline. The WSC validator predicates imply the stated coverage claim, and the margins quantify robustness under the declared perturbation model.
- **Obstruction.** A witness obstruction (WOB) whose code belongs to the finite catalog and whose evidence satisfies that code’s checklist. Within the declared regime, this evidence rules out the existence of any WSC satisfying the same regime validators for the stated target.

Proof skeleton (predicate exhaustion)

The completeness content is carried by a finite suite of validator predicates. Each predicate corresponds to one canonical obstruction code, and each predicate failure forces its code. Within the declared regime, any failed validator yields a WOB whose evidence is auditable. If all validators hold, then the forcing templates and target-correctness rules permit a WSC.

Predicate exhaustion: minimal completeness skeleton

SFT treats *validator predicates* as the formal boundary between certificate-claims and obstruction-claims.

The audit-facing specification of the predicate suite and its canonical mapping appears in `VALIDATOR_PREDICATE_SPEC.tex` (and in the runnable contract files under `engine/contracts`). Inside the declared regime, a witness stability certificate (WSC) is permitted only when the listed predicates hold on the declared target. When a predicate fails, the correct output is a witness obstruction (WOB) with a canonical code and a finite evidence checklist.

Validator predicates and canonical WOB codes

Write V_{XX} for the validator predicate associated to obstruction code **WOB-XX**. The intended reading is:

- if V_{XX} holds, then **WOB-XX** is *not* triggered by that failure mode,
- if V_{XX} fails, then **WOB-XX** is the correct obstruction code to report (with evidence).

The predicates are:

- **V_{01} (subsystem/observable present)**. There exists an admissible subsystem (slice, orbit-domain, or embedded loop/defect domain) and a phase observable to a declared target $\mathbb{T}_P$ or $\mathbb{T}_P$. If this fails, report **WOB-01-no-subsystem**.
- **V_{02} (nontrivial winding/holonomy)**. In an embedded mechanism, some certified loop carries a nonzero integer witness (degree or holonomy multiple). If this fails, report **WOB-02-trivial-winding**.
- **V_{03} (no pinch on the declared target)**. On the declared target, the observable image does not miss a positive-length arc neighborhood required for SCL. If this fails, report **WOB-03-pinch**.
- **V_{04} (no tear: definability/regularity)**. The hypotheses needed to define degree/holonomy in the adopted model class hold on the evidence domain (liftability/atlas consistency/continuity at the declared tolerance). If this fails, report **WOB-04-tear**.
- **V_{05} (holonomy consistency)**. In holonomy-based mechanisms, the lift/connection interface is consistent at the declared tolerance so the holonomy integer is well-defined. If this fails, report **WOB-05-holonomy-defect**.
- **V_{06} (coupled target correctness)**. In coupled settings, the selected target is the coupled subtorus coset CST_s and the baseline is coupled Haar on CST_s . If this fails, report **WOB-06-coupling-mismatch**.
- **V_{07} (mixing claim justified)**. Any statistical convergence claim is accompanied by an admissible certificate hypothesis (RNF or NDB+MXC, with declared parameters). If this fails, report **WOB-07-mixing-not-justified**.
- **V_{08} (orbit realized on the evidence domain)**. In action-equivariant mechanisms, the evidence domain contains a χ -active circle-type orbit segment needed for the forcing template. If this fails, report **WOB-08-orbit-collapse**.
- **V_{09} (equivariance within tolerance)**. The equivariance defect is within the declared admissible tolerance on the evidence domain. If this fails, report **WOB-09-equivariance-defect**.
- **V_{10} (no resonant locking for the intended target)**. The response rank and the inferred constraint structure do not collapse the intended target (full torus versus subtorus coset). If this fails, report **WOB-10-resonant-locking**.
- **V_{11} (connected slice/component availability)**. The declared measurement domain intersects the orbit-bearing/loop-bearing component required by the chosen mechanism. If this fails, report **WOB-11-disconnected-slice**.

- **V₁₂ (slope-zero quotient surjectivity when needed).** When the induced homomorphism χ is trivial, the factor map through the quotient is surjective onto the declared target. If this fails, report WOB-12-slope-zero-quotient-nonsurjective.
- **V₁₃ (constraint hypothesis uniqueness).** In constraint/target inference, the declared data and tolerance determine a unique small-integer constraint hypothesis within the declared finite search. If this fails, report extttWOB-13-ambiguity.
- **V₁₄ (sufficient excitation for inference).** The sample count and excitation are sufficient to support stable inference of constraint rank and target rank at the declared tolerance. If this fails, report extttWOB-14-insufficient-excitation.
- **V₁₅ (stability margin away from resonance).** The best constraint candidates are separated from the declared tolerance by a positive stability margin; near-relations do not make the decision unstable. If this fails, report extttWOB-15-resonance-masking.

Lemma suite (each predicate failure forces its WOB code)

Lemma 1 (WOB-01 is forced by $\neg V_{01}$). *If no admissible subsystem/observable exists in the declared regime, then no WSC exists in that regime.*

Proof. Every WSC is defined on a declared evidence domain and records a target and observable. If such data do not exist in the declared regime, then no WSC object satisfying the regime validators can exist. $\square$

Lemma 2 (WOB-02 is forced by $\neg V_{02}$). *If every admissible loop mechanism in the evidence domain has zero degree and zero holonomy multiple, then no embedded-loop WSC with nonzero witness integer exists.*

Proof. Embedded-loop WSCs require an integer witness invariant that persists under the LCM/HMR margins. If all admissible loops have witness integer 0, then the mechanism cannot certify forced coverage by winding/holonomy and the obstruction is the trivial-winding class. $\square$

Lemma 3 (WOB-03 is forced by $\neg V_{03}$). *If the observable image misses a required arc neighborhood on the declared target, then SCL on that target cannot hold, hence no target-correct WSC exists.*

Proof. SCL asserts surjectivity onto the declared target. Missing an arc neighborhood contradicts surjectivity, so no WSC proving SCL on that target can exist. $\square$

Lemma 4 (WOB-04 is forced by $\neg V_{04}$). *If the adopted model class does not provide the regularity needed to define the witness invariant (degree/holonomy) robustly at the declared tolerance, then no WSC with certified margins exists.*

Proof. The WSC format requires a well-defined integer witness together with margins certifying its stability. If the witness cannot be defined in the adopted model class on the evidence domain, the certificate interface cannot be satisfied. $\square$

Lemma 5 (WOB-05 is forced by $\neg V_{05}$). *If holonomy consistency fails at the declared tolerance, then holonomy-based certification is ill-posed in that regime and no holonomy WSC exists.*

Proof. Holonomy WSCs require a connection/lift interface whose loop integral is well-defined modulo the period. If the interface is inconsistent, the holonomy integer is not stable or not defined, and the holonomy-defect obstruction is the correct output. $\square$

Lemma 6 (WOB-06 is forced by $\neg V_{06}$). *In a coupled setting, if the target is not declared as CST_s with coupled Haar baseline, then any claimed certificate is target-incorrect and no target-correct WSC exists without repairing the target.*

Proof. Coupling changes the maximal invariant set and the correct baseline measure. A certificate that uses the wrong target/baseline is not a WSC in the declared target-correct regime. $\square$

Lemma 7 (WOB-07 is forced by $\neg V_{07}$). *If a statistical convergence claim is asserted without an admissible hypothesis certificate (RNF or NDB+MXC), then the claim is outside the permitted regime and the correct output is the mixing-not-justified obstruction.*

Proof. SFT treats convergence as an additional artifact class whose hypotheses must be declared and auditable. Without an admissible hypothesis certificate, no convergence certificate can be issued, so the regime boundary is triggered. $\square$

Lemma 8 (WOB-08 is forced by $\neg V_{08}$). *If the χ -active orbit needed by the equivariant forcing template is absent from the evidence domain, then action-equivariant forcing cannot be invoked there and no forcing-based WSC exists on that domain.*

Proof. The forcing template certifies coverage by transporting phases along a realized orbit. If no such orbit occurs on the evidence domain, the mechanism is unavailable and the orbit-collapse class is the correct obstruction. $\square$

Lemma 9 (WOB-09 is forced by $\neg V_{09}$). *If equivariance defect exceeds the declared tolerance, then equivariance is not valid in the adopted model class at that tolerance and no equivariance-based WSC exists.*

Proof. Equivariance-based WSCs require the defect to be within tolerance so that the induced homomorphism controls orbit images. If the defect is too large, the required mechanism hypotheses fail. $\square$

Lemma 10 (WOB-10 is forced by $\neg V_{10}$). *If rank deficiency or resonant locking collapses the intended target, then no certificate can truthfully assert coverage of the larger target; the correct output is the resonant-locking obstruction (and the target must be revised).*

Proof. The correct target is determined by the response rank and by any valid integer constraints. If the intended target is larger than the reachable coset/subtorus, coverage of the larger target is impossible. $\square$

Lemma 11 (WOB-11 is forced by $\neg V_{11}$). *If the declared measurement domain does not meet the required orbit-bearing/loop-bearing component, then no certificate for that mechanism can be produced from that domain.*

Proof. All certificate mechanisms require evidence on a domain supporting the relevant orbit or loop. If the domain misses that component, the mechanism is inapplicable there. $\square$

Lemma 12 (WOB-12 is forced by $\neg V_{12}$). *If χ is trivial and the quotient factor map is not surjective onto the declared target, then SCL on that target is impossible and no target-correct WSC exists.*

Proof. When χ is trivial, equivariance forces the observable to factor through the quotient. If the induced factor is not surjective, the declared target cannot be covered. $\square$

Lemma 13 (extttWOB-13 is forced by egV_{13}). *If multiple incompatible small-integer constraint hypotheses fit within the declared tolerance and search, then the target is not uniquely determined in the declared regime and no target-correct certificate can be issued without additional excitation or a refined target declaration.*

Proof. A target-correct certificate requires a declared target type and, when constraints are part of the target selection, a determinate constraint hypothesis. If the declared data/tolerance admit more than one incompatible constraint hypothesis within the search, any choice is non-auditable without additional information, so the declared regime treats this as an explicit inference obstruction. $\square$

Lemma 14 (extttWOB-14 is forced by egV_{14}). *If the sample count or excitation is insufficient to support stable constraint inference at the declared tolerance, then inference is ill-posed in the declared regime and the correct output is extttWOB-14-insufficient-excitation.*

Proof. Stable inference requires enough samples and enough motion to distinguish candidate constraint hypotheses. When these are absent, any inferred constraints can change under small perturbations of the data, so no target-correct certificate can be issued in the declared regime. $\square$

Lemma 15 (extttWOB-15 is forced by egV_{15}). *If the best candidate constraints are within the declared tolerance by too small a margin, or if a near-relation lies just above tolerance, then the constraint decision is unstable at the declared tolerance and the correct output is extttWOB-15-resonance-masking.*

Proof. A positive stability margin is needed so that the inferred constraints persist under admissible perturbations. When the margin is too small or a near-relation is present, small perturbations can change the inferred target rank or constraint set, so the declared regime treats the situation as a resonance-masking obstruction. $\square$

Validator predicate specification (audit interface)

The completeness claim in the declared regime is enforced by a finite suite of *validator predicates*. Each predicate V_{XX} corresponds to one canonical obstruction code WOB-XX. For an issued WSC, the applicable predicates must hold; otherwise the correct output is a WOB with the associated code.

For audit-level synchronization between the paper and the runnable artifacts, the normative mapping is also recorded in:

- engine/contracts/VALIDATOR_PREDICATE_SPEC.md
- engine/contracts/validator_predicates.json
- engine/contracts/wob_codebook.json

Pred.	Canonical WOB code	Applicability (minimal)	Audit hook (artifact-level)
V01	WOB-01-no-subsystem	always	audit_wsc.py: requires a nonempty target and a predicate ledger entry for V01
V02	WOB-02-trivial-winding	mechanisms LDW/HW	audit_wsc.py: requires witness integer $k \neq 0$ and a V02 ledger entry

Pred.	Canonical WOB code	Applicability (minimal)	Audit hook (artifact-level)
V03	WOB-03-pinch	when arc-neighborhood coverage is asserted	recorded as an evidence reference (no recomputation in v0.1)
V04	WOB-04-tear	mechanisms LDW/HW	<code>audit_wsc.py</code> : requires declared perturbation model + margins and a V04 ledger entry
V05	WOB-05-holonomy-defect	when holonomy multiple is invoked	<code>audit_wsc.py</code> : requires <code>margins.HMR_P</code> and a V05 ledger entry
V06	WOB-06-coupling-mismatch	target is CST (or mechanism CMT)	CTP/CHT (coupling target correctness) enforced: requires <code>target_type=cst</code> and baseline declaring coupled Haar on CST
V07	WOB-07-mixing-not-justified	mechanism MXC	<code>audit_wsc.py</code> : requires a convergence certificate (RNF or NDB+MXC) and a V07 ledger entry
V08	WOB-08-orbit-collapse	mechanism AWT	<code>audit_wsc.py</code> : requires a V08 ledger entry with evidence reference
V09	WOB-09-equivariance-defect	mechanism AWT	<code>audit_wsc.py</code> : requires a V09 ledger entry with evidence reference
V10	WOB-10-resonant-locking	when full-torus target is asserted	recorded as an evidence reference (rank/constraint inference expands in SFT III)
V11	WOB-11-disconnected-slice	when component restriction is used	recorded as an evidence reference (connectedness tests expand with payloads)
V12	WOB-12-slope-zero-quotient-nonsurjective	when χ is trivial and quotient is invoked	recorded as an evidence reference (quotient surjectivity expands with payloads)
V13	WOB-13-ambiguity	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)
V14	WOB-14-insufficient-excitation	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)

Pred.	Canonical WOB code	Applicability (minimal)	Audit hook (artifact-level)
V15	WOB-15-resonance-masking	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)

The WSC artifact format includes a lightweight predicate ledger field `validator_predicates` that records the applicable predicates as “holds” with short evidence references. The audit layer checks the parts that are readable from the artifact interface and rejects target-incorrect coupled claims.

Finite obstruction list is complete in the declared regime

Inside the declared regime, the obstruction interface is not an open-ended list of failure stories. It is a finite catalog whose members are forced by validator predicate failures. This is the sense in which the catalog is *complete*: every certificate failure inside the regime is recognized as exactly one canonical obstruction code with auditable evidence.

Canonical obstruction catalog (WOB)

WOB code	Meaning (declared regime)
WOB-01-no-subsystem	No admissible cyclic subsystem/observable is available in the measured model.
WOB-02-trivial-winding	All candidate loops have degree/holonomy multiple 0.
WOB-03-pinch	Image misses an arc neighborhood on the declared target (non-surjective).
WOB-04-tear	Regularity/definability needed for degree/holonomy fails at the declared tolerance.
WOB-05-holonomy-defect	Lift/connection consistency needed for holonomy certification fails.
WOB-06-coupling-mismatch	Coupling target/baseline is wrong (CST not computed or baseline not defined).
WOB-07-mixing-not-justified	Statistical convergence claim made without an admissible RNF or NDB+MXC hypothesis.
WOB-08-orbit-collapse	No χ -active circle-type orbit exists on the declared evidence domain.
WOB-09-equivariance-defect	Equivariance defect exceeds the declared admissible tolerance.
WOB-10-resonant-locking	Rank deficiency/rational dependence collapses the intended torus target.
WOB-11-disconnected-slice	Measurement domain misses the orbit/loop component required by the mechanism.
WOB-12-slope-zero-quotient-nonsurjective	χ is trivial and the induced quotient factor map is not surjective.

Theorem (catalog completeness within the declared regime)

Fix the declared regime described above (declared target type, admissible mechanism, admissible FEQ class, declared tolerances and margins, and in coupled settings the target-correctness rule selecting CST_s with coupled Haar baseline). Let WSC mean: *there exists a witness stability certificate satisfying the regime validators on the declared correct target*. Let WOB mean: *there exists a witness obstruction whose code is one of the fifteen canonical codes and whose evidence satisfies its checklist*. Then, inside the declared regime,

$$\neg WSC \iff WOB.$$

More concretely: a validator predicate failure $\neg V_{XX}$ forces the obstruction code $WOB-XX$, and if all validator predicates hold then the regime permits a WSC on the correct target.

Necessity (each obstruction code is realized by an audited atlas artifact)

Each canonical WOB code occurs in the shipped witness atlas as a boundary artifact with concrete evidence. The following obstruction artifacts are included under `engine/artifacts/` and are checked by the audit suite:

WOB code	Realizing atlas obstruction artifact
WOB-01-no-subsystem	WOB-ATLAS-0013.json
WOB-02-trivial-winding	WOB-ATLAS-0014.json
WOB-03-pinch	WOB-ATLAS-0015.json
WOB-04-tear	WOB-ATLAS-0016.json
WOB-05-holonomy-defect	WOB-ATLAS-0031.json
WOB-06-coupling-mismatch	WOB-ATLAS-0011.json
WOB-07-mixing-not-justified	WOB-ATLAS-0020.json
WOB-08-orbit-collapse	WOB-ATLAS-0019.json
WOB-09-equivariance-defect	WOB-ATLAS-0012.json
WOB-10-resonant-locking	WOB-ATLAS-0009.json
WOB-11-disconnected-slice	WOB-ATLAS-0017.json
WOB-12-slope-zero-quotient-nonsu rjective	WOB-ATLAS-0018.json

This realizability statement is what makes the catalog *necessary*: no code can be removed without deleting a genuine, audited boundary mechanism.

Corollary (well-posed universality questions)

A universality question is well-posed precisely when its target and hypotheses are stated in a form that lies inside the declared regime, so that the engine can return either a WSC or a WOB on the correct target.

The Syncre Stability Modulus

To give SFT a single intrinsic “state” variable that can be compared across systems, we package the operational ingredients of stability and convergence into one modulus.

Definition (Syncre Stability Modulus)

On a declared target (full torus or coupled target), define the Syncre Stability Modulus

$$\mathfrak{S} = (k, \text{LCM}, \text{HMR}, b, r, \lambda),$$

where:

- k is the witness integer (degree or holonomy multiple).
- LCM is the lift-closeness margin guaranteeing degree stability.
- HMR is the holonomy margin guaranteeing holonomy class stability.
- b is a distortion/branch bound used for quantitative anti-collapse on a witness loop (when applicable).
- r is the effective target rank (full torus rank or free-coordinate rank on CST_s).
- λ is a certified convergence rate when a diffusion/mixing model is adopted (otherwise λ is left unspecified).

How the modulus is used

SFT statements can be phrased as threshold claims on $\mathfrak{S}$:

- **Existence and persistence.** If $k \neq 0$ and $\text{LCM} < \pi$ (and/or $\text{HMR} < P/2$), then the witness tier persists under the perturbation model.
- **Quantitative non-collapse.** If the witness loop satisfies a monotone-branch structure and branch bound b , then the coverage profile admits explicit lower bounds proportional to $(|k|/b)$.
- **Convergence when justified.** If a diffusion/noise bridge is adopted and the mixing certificate holds, then λ is read directly from the thresholds ledger and recorded as part of the certificate.

The modulus is not a new physical constant. It is a compact way to record what SFT needs in order to certify inevitability, stability, and (when appropriate) convergence.

Modulus laws, composition, and sharp failure

The stability modulus $\mathfrak{S}$ is the native invariant for the engine.

Theorem 1 (FEQ invariance). *If two witness contexts are frame-equivalent (FEQ) and the witness integer k and declared margins (LCM/HMR) coincide, then the engine outputs are equivalent up to the same FEQ transformation: a certificate remains in the same certificate class and carries the same modulus $\mathfrak{S}$.*

Theorem 2 (Margin crossing forces obstruction). *Fix a perturbation model. If a perturbation exceeds the declared margin recorded in $\mathfrak{S}$ (for example, $\text{LCM} \geq \pi$ for degree stability or $\text{HMR} \geq P/2$ for holonomy stability), then the correct engine output is not a modified certificate: it is a boundary report. In artifact form this is a WOB whose code matches the first failed validator predicate.*

Theorem 3 (Coupled targets are meaning). *If coupling restricts the image to a coupled subtorus CST_s , then any statistical baseline on the full torus is ill-posed. In artifact form, the validator predicate V06 enforces that the baseline is coupled Haar on CST_s , and failure produces WOB-06-coupling-mismatch.*

Composition and degradation (artifact rules)

Certificate composition is treated as an explicit transformation on artifacts.

- Composition rules are listed in `engine/contracts/certificate_composition_rules.md`.
- Degradation rules and their audit interface are listed in `engine/contracts/certificate_degradation_rules.md`.

When a certificate declares a degradation inequality, the audit script verifies it numerically.

Modulus stability law (summary)

When a certificate records the Synchré Stability Modulus $\mathfrak{S}$, its meaning is operational: within the declared perturbation model, the witness integer cannot change unless a stability margin is crossed. Under frame equivalence (FEQ) and dynamics preserving the recorded invariants, the certificate class is unchanged. When margins are crossed or assumptions fail, the correct output is an obstruction report rather than a modified certificate.

Classification program: target type and obstruction type

SFT organizes cyclic phase-field systems by two coordinates:

- **Target type:** full circle, full torus T^k , or coupled target CST_s .
- **Obstruction type:** one of the finite WOB classes when a witness cannot be certified.

Class	Meaning in SFT
Class A: forced full coverage	WSC exists on full circle or full torus; stability margins are recorded; diagnostics target Haar baseline.
Class B: forced coupled coverage	WSC exists on CST_s ; baseline is coupled Haar on CST_s ; convergence statements target coupled Haar.
Class C: embedded-only coverage	WSC exists on an embedded subsystem (loop/defect/orbit) even if global coverage is not claimed.
Class D: no forced coverage in declared regime	WOB exists; the obstruction code identifies why cyclic forcing cannot be certified in the declared measurement model.

This classification is the “periodic table” of SFT: different-looking systems become the same object once the target type and obstruction type are identified.

Canonical questions SFT answers

SFT is designed to answer a fixed set of questions better than ad hoc reasoning:

- **What is the correct target.** Is the intended target the full circle/torus, or does coupling force a coupled target CST_s .

- **Is coverage forced.** Do the symmetry-and-nondegeneracy inputs force full coverage on the correct target (globally or on an embedded subsystem).
- **Is the witness stable.** Under the declared perturbation model, do explicit margins certify persistence of the witness.
- **If coverage fails, why.** Which finite obstruction class occurs, and what evidence supports it.
- **If convergence is claimed, what certifies it.** Is there a nonresonance certificate or a diffusion/mixing certificate, and what baseline and rate are justified.

The engine posture is that each question has an output: either a certificate with margins on the correct target, or a named obstruction with evidence.

Canonical diagnostics and baseline selection

SFT uses a small set of diagnostics. The baseline measure is chosen by a single rule:

Baseline selection rule. If coupling constrains the image to a coupled subtorus coset CST_s , the baseline is coupled Haar on CST_s . Otherwise the baseline is Haar on the full torus.

Diagnostic	Definition	Target
Coverage profile (CP)	$\text{CP}(\rho) = \inf_y \mu(B_\rho(y))$ for the pushforward $\mu = \Phi_\# \nu$ and arc neighborhood $B_\rho(y)$	Haar on T_P or coupled Haar on CST_s
Fourier uniformity defect (FUD)	$\text{FUD}_N = \sum_{1 \leq n \leq N} \hat{p}(n) $ for phase density p on $\mathbb{T}_P$	Haar baseline (uncoupled) or free-coordinate Haar (coupled)
Spatial uniformity defect (SUD)	distance of μ_s from the baseline measure in a chosen metric (TV or L^2 when available)	baseline measure per rule above

Rates and thresholds used with these diagnostics are centralized in `engine/contracts/constants_thresholds.md`.

Obstruction table (dichotomy support for universality)

SFT treats coverage claims as engine outputs: either a WSC (witness stability certificate) exists, or a WOB (witness obstruction) exists. The table below provides the canonical mapping used in the paper.

Obstruction	WOB code	Evidence checklist pointer
No cyclic subsystem / no phase observable	WOB-01-no-subsystem	<code>engine/contracts/WOB_CODEBOOK.md</code>

Obstruction	WOB code	Evidence checklist pointer
Trivial winding / holonomy multiple zero	WOB-02-trivial-winding	engine/contracts/WOB_CODEBOOK.md
Image misses an arc (pinch)	WOB-03-pinch	engine/contracts/WOB_CODEBOOK.md
Regularity failure (tear)	WOB-04-tear	engine/contracts/WOB_CODEBOOK.md
Holonomy certification defect	WOB-05-holonomy-defect	engine/contracts/WOB_CODEBOOK.md
Coupling target not computed or wrong baseline	WOB-06-coupling-mismatch	engine/contracts/WOB_CODEBOOK.md
Mixing claim made without NDB/MXC	WOB-07-mixing-not-justified	engine/contracts/WOB_CODEBOOK.md
No χ -active circle-type orbit (orbit collapse)	WOB-08-orbit-collapse	engine/contracts/WOB_CODEBOOK.md
Equivariance defect beyond tolerance	WOB-09-equivariance-defect	engine/contracts/WOB_CODEBOOK.md
Rank deficiency / resonant locking	WOB-10-resonant-locking	engine/contracts/WOB_CODEBOOK.md
Disconnected slice misses witness component	WOB-11-disconnected-slice	engine/contracts/WOB_CODEBOOK.md
Slope-zero plus quotient nonsurjectivity	WOB-12-slope-zero-quotient-nonsurjective	engine/contracts/WOB_CODEBOOK.md

2 Contribution summary table

Component	Standard ingredient	SFT contribution (what is added)
Witness forcing	Degree, winding, equivariance	Witness engine tiers (AWT/LDW/HW/CMT) expressed as auditable artifacts with checklists and margins
Coupling and targets	Subtori and cosets are classical	Target correctness as a rule: CST targets and coupled Haar baselines are enforced by predicates (CTP/CHT)
Exhaustiveness	Logical dichotomies are classical	Predicate-exhaustion exhaustiveness: inside COC, every declared claim resolves to WSC or first-failure WOB code
Stability	Continuity of invariants is classical	Stability margins (LCM/HMR and derived modulus fields) stored as first-class certificate data
Target inference	Integer relations are classical	Conservative CST inference: bounded small-integer constraints with explicit stability gaps or obstruction codes

Component	Standard ingredient	SFT contribution (what is added)
Diagnostics	Pushforward measures are classical	Target-aligned diagnostics (coverage profile and uniformity defects) attached to correct targets
Auditability	Not standard in theory papers	Schemas, audits, benchmarks, and hash-verified builds tying theorems to checkable objects

3 Related work and positioning

SFT is built from standard ingredients:

- **Degree and winding.** Coverage forcing on a circle target is governed by degree of loop maps $S^1 \rightarrow S^1$ and homotopy invariance [1, 2].
- **Compact group actions and equivariance.** Action witnesses are compact transformation group arguments: equivariant observables reduce forcing to the induced circle homomorphism and orbit structure [3].
- **Torus translations and equidistribution.** When convergence to Haar is asserted, background facts include unique ergodicity of irrational torus translations and standard equidistribution criteria [4, 6, 7].
- **Noise, diffusion, and mixing.** Quantitative convergence in the presence of noise is modeled via Markov or diffusion operators on compact manifolds under explicit spectral-gap or minorization conditions [8, 9, 17, 18].
- **Integer relations and constraint discovery.** Coupled-target inference uses bounded small-integer relation search or lattice reduction methods; stability is expressed through explicit margins [15, 19].

What SFT adds

SFT contributes a verifiable contract for phase-coverage statements rather than a new topological degree theorem. The distinctive content is the target-correct, audited interface:

- **Correct-target discipline.** Coupling changes the target and the baseline; certificates are rejected when they silently use Haar on the ambient product torus while the correct target is a coupled coset.
- **Certificate-or-obstruction decision objects.** Every declared claim in COC resolves to a finite object: WSC with margins or WOB with a canonical code and evidence pointers.
- **Predicate-exhaustion exhaustiveness.** The finite validator predicate suite makes refusal modes explicit and exhaustive inside COC.
- **Reproducible artifact bundle.** Theorems are linked to schemas, audits, benchmarks, and hash-verified builds so readers can independently validate claims.

4 Definitions: sliced arenas and the Syncré Coverage Law

Fix $P > 0$ and $\mathbb{T}_P := \mathbb{R}/P\mathbb{Z}$ [2]. A *sliced arena* is a decomposition

$$M = \bigsqcup_{s \in \mathbb{R}} \Sigma_s,$$

with slice parameter s representing a chosen global-instant labeling.

A *phase field* is a map $\Phi : M \rightarrow \mathbb{T}_P$. Write $\Phi_s := \Phi|_{\Sigma_s}$.

SCL (Syncré Coverage Law)

$$\boxed{\forall s \in \mathbb{R}, \quad \Phi(\Sigma_s) = \mathbb{T}_P.}$$

SLL (Syncré Lift Law)

A lift interface is specified by $\Phi = F \bmod P$ for a real-valued lift F on the relevant domain (globally or via a lift atlas), enabling nonlinear deformation while keeping analysis in $\mathbb{R}$.

5 Witness engines: action and loop mechanisms

SFT is witness-first: coverage claims are supported by finite witness data.

5.1 Slice-level action witness template (AWT)

If a slice Σ carries a continuous S^1 -action $A : S^1 \times \Sigma \rightarrow \Sigma$ and there exists an equivariant phase observable $\phi : \Sigma \rightarrow \mathbb{T}_P$ with nonzero-slope phase response $\chi : S^1 \rightarrow \mathbb{T}_P$ [3],

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta), \quad \chi(S^1) = \mathbb{T}_P,$$

then equivariance forces stabilizers into the kernel of χ , so in the nonzero-slope regime the action is circle-type on the forcing domain. In particular, $\phi(\Sigma) = \mathbb{T}_P$.

5.2 Embedded witnesses (ESW)

Many realizations expose phase structure on embedded subsystems. An embedded Syncré witness is an embedding $i : E \hookrightarrow \Sigma_s$ such that $\Phi \circ i(E) = \mathbb{T}_P$. For $E \cong S^1$, this is certified by:

- **LDW (Loop degree witness).** $\deg(\Phi \circ \gamma) \neq 0$ for a loop $\gamma : S^1 \rightarrow \Sigma_s$ [1].
- **HW (Holonomy witness).** With a connection form α (from a lift atlas), $\int_\gamma \alpha = mP$ with $m \neq 0$.

6 Universality (scope and meaning)

Universality inside COC

SFT does not assert that every system satisfies slice-wise coverage. Instead, SFT fixes a named regime (COC) where the question has a closed, auditable meaning: a declared phase observable, a declared slice/subsystem, a target rule (including coupling conventions), and an admissible witness mechanism.

Within COC, the engine returns either a WSC proving coverage on the correct target or a WOB naming a finite obstruction code with evidence.

Universality-from-cyclic-structure schema

Definition 2 (Cyclic Observability Class (COC)). *The Cyclic Observability Class (COC) is the class of declared SFT contexts c satisfying all of the following three conditions (and it is maximal by definition, since membership is exactly these conditions):*

- *a phase observable is declared on the relevant slice/subsystem: a cycle-valued map (circle/torus/coupled target) with the minimal regularity needed to define the intended winding/holonomy notion and to test nontriviality;*
- *an admissible witness mechanism is declared (symmetry action, loop/holonomy, embedded subsystem witness, defect loop, or another explicitly listed mechanism), so the engine has a prescribed route to either a WSC or to a named obstruction code;*
- *a correct target rule is satisfied: the target type (circle/torus/CST) and any baseline statements are defined on the correct target (Haar on the target, coupled Haar on CST when coupling is present).*

Definition 3 (Universality on a class of contexts). *Fix a class of contexts $\mathcal{C}$. The SFT engine is universal on $\mathcal{C}$ if for every declared context $c \in \mathcal{C}$ and every slice/subsystem where the claim is posed, the engine returns exactly one of:*

- *a Witness Stability Certificate (WSC) proving coverage on the correct target in c (with explicit margins), or*
- *a Witness Obstruction (WOB) belonging to a finite obstruction catalog (with evidence), certifying that the claim is blocked in c .*

Global universality means: universal on COC. In SFT, a statement labeled *global universality* is a universality statement on COC: for every context in COC, the engine returns a WSC or a WOB on the correct target; outside COC, claims are either inapplicable (target/predicates not well-posed) or produce a boundary-type obstruction rather than a forced-coverage theorem.

COC checklist (entrywise). A declared context is treated as belonging to COC only when all of the following are explicitly provided:

- **Phase observable (PO).** A named cycle-valued observable on the intended slice/subsystem (circle/torus/CST), together with the minimum regularity assumptions needed to define its winding/holonomy notion and to test nontriviality.
- **Admissible witness mechanism (WM).** A declared mechanism from the admissible menu (symmetry action, loop/holonomy, embedded subsystem witness, defect loop, or another explicitly listed mechanism) that specifies what the engine must attempt to certify.
- **Target rule (TR).** A declared target type (circle/torus/CST) and baseline convention on that target (Haar on the target; coupled Haar on CST when coupling is present).

When any checklist item fails, the correct engine output is a named obstruction (WOB) rather than a coverage certificate.

Theorem 4 (Universality-from-cyclic-structure (COC embedding schema)). *Fix a declared model class and let Σ denote the slice/subsystem on which a Synchré claim is posed. Assume the declaration provides:*

- **Phase observable (PO).** A cycle-valued observable $\phi : \Sigma \rightarrow \mathbb{T}_P$ or $\Phi : \Sigma \rightarrow \mathbb{T}_P$ with the minimal regularity needed to define the intended winding/holonomy notion and to test nontriviality.
- **Witness mechanism (WM).** At least one admissible mechanism from the SFT menu, in one of the following structural forms:
 - Action forcing. An admissible continuous action $A : G \times D \rightarrow D$ on a domain $D \subseteq \Sigma$ with $G = S^1$ or $G = \mathbb{T}^k$, together with a continuous homomorphism $\chi : G \rightarrow H$ to the declared target group H such that ϕ or Φ is χ -equivariant on D in the declared tolerance (exactly or with a stated defect bound). The action-forcing mechanism is nondegenerate when the induced map on the declared target has nontrivial cyclic response (nonzero slope in the S^1 case; positive rank on the target coordinates in the $\mathbb{T}^k$ case).
 - Embedded loop/holonomy. An embedded loop $\gamma : S^1 \rightarrow \Sigma$ (or loop in an embedded subsystem) together with either a nonzero degree witness $\deg(\phi \circ \gamma) \neq 0$ or a nonzero holonomy multiple in a declared lift/connection model.
- **Target rule (TR).** A declared target type (circle/torus/coupled subtorus) and baseline convention on that target (Haar on the target, coupled Haar on CST when coupling is present).

Then the declared context belongs to COC. Consequently, the SFT universality posture applies on this context: on the declared target, the correct engine output is either a Witness Stability Certificate (WSC) with explicit margins, or a finite Witness Obstruction (WOB) from the obstruction catalog with evidence.

Canonical cyclic families covered by the schema

The theorem is a *schema*: it reduces broad families to the same three-checklist declaration (PO/WM/TR).

The circle-action / periodic-forcing family admits a fully formal proof template in Theorem 5. Each line below is a structural criterion (not a physical promise): if the criterion holds in the adopted model class, it produces a COC embedding.

- **Periodic forcing / time-shift symmetry.** If the declared subsystem admits a continuous periodic parameter (identified with S^1) acting by reparameterization on states, and the phase observable is equivariant under that shift with nonzero slope on the declared phase circle, then the action-forcing mechanism applies.
- **Rotation / rigid cyclic response.** If the declared subsystem carries an admissible S^1 -action (or a torus action in the multi-cycle setting) and the observable is equivariant with nontrivial induced homomorphism on the declared target, then forcing applies on the corresponding orbit-image (full circle when slope is nonzero; subtorus/coset when rank is deficient and the target rule selects CST).
- **Wave-train / translation-phase reduction.** If a periodic direction induces an admissible S^1 -translation action on a domain and the phase observable is equivariant under translation with nonzero slope, the situation is again an action-forcing context.
- **Limit-cycle / periodic-orbit phase.** If the declared dynamics exhibits a periodic orbit equipped with a continuous phase coordinate (so that motion along the orbit defines an S^1 -action on that orbit) and the observable responds with nonzero winding along that cycle, then either action-forcing or an embedded loop degree witness certifies cyclic observability.

- **Defect loop / winding witness.** If the declaration provides a loop around a distinguished locus (a boundary component, defect core, or excluded set) on which the phase observable has nonzero winding (or nonzero holonomy multiple in a lift model), then the embedded witness mechanism applies.
- **Coupled drives / multi-cycle response.** If multiple cyclic inputs generate an admissible $\mathbb{T}^k$ -action and the observable is equivariant with integer-matrix response χ , then the correct target is the image subtorus (or the inferred coupled subtorus CST), and the forcing/obstruction dichotomy is posed on that target.

Canonical proof template: circle-action forcing (rotation / periodic forcing)

Theorem 5 (Circle-action forcing template). *Let Σ be a slice and let $A : S^1 \times \Sigma \rightarrow \Sigma$ be a continuous S^1 -action. Let $\phi : \Sigma \rightarrow \mathbb{T}_P$ be continuous. Assume there exists a continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P$ such that on a declared domain $D \subseteq \Sigma$,*

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) \quad \forall (\theta, x) \in S^1 \times D. \quad (1)$$

Write $\deg(\chi) \in \mathbb{Z}$ for the integer degree of χ .

- If $\deg(\chi) \neq 0$, then for every $x \in D$ the orbit loop $\gamma_x : S^1 \rightarrow \Sigma$, $\gamma_x(\theta) = A(\theta, x)$ satisfies

$$\deg(\phi \circ \gamma_x) = \deg(\chi) \neq 0,$$

and $\phi(D) = \mathbb{T}_P$. In particular, the declaration (Σ, ϕ) satisfies the COC checklist on D with: (PO) phase observable ϕ , (WM) action forcing and embedded loop witness, (TR) circle target with Haar baseline.

- If $\deg(\chi) = 0$, then ϕ is S^1 -invariant on D and factors through the orbit quotient $q : D \rightarrow D/S^1$: there exists $\bar{\phi} : D/S^1 \rightarrow \mathbb{T}_P$ with $\phi|_D = \bar{\phi} \circ q$. In this case, coverage on $\mathbb{T}_P$ can occur only through surjectivity of $\bar{\phi}$.

Proof. Every continuous homomorphism $\chi : S^1 \rightarrow S^1$ has the form [3, 1]: $\chi(e^{2\pi it}) = e^{2\pi imt}$ for a unique $m \in \mathbb{Z}$, and $\deg(\chi) = m$.

Fix $x \in D$. Define $\gamma_x(\theta) = A(\theta, x)$. By (1),

$$(\phi \circ \gamma_x)(\theta) = \phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta).$$

Translation by the constant $\phi(x)$ does not change degree [1], hence $\deg(\phi \circ \gamma_x) = \deg(\chi)$.

If $\deg(\chi) \neq 0$, then χ is surjective onto $\mathbb{T}_P$. For any $y \in \mathbb{T}_P$, choose $\theta \in S^1$ with $\chi(\theta) = y \ominus \phi(x)$. Then $\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) = y$, so $\phi(D) = \mathbb{T}_P$. The nonzero degree identity above supplies an embedded loop/holonomy witness, and the equivariance identity supplies an admissible action-forcing witness mechanism.

If $\deg(\chi) = 0$, then χ is trivial, so (1) gives $\phi(A(\theta, x)) = \phi(x)$ for all θ and all $x \in D$. Thus ϕ is constant along orbits in D and therefore factors through the orbit space [3]: there exists $\bar{\phi}$ with $\phi|_D = \bar{\phi} \circ q$. $\square$

Theorem 6 (Periodic forcing / rotation model as a COC embedding). *Let Σ be a slice in a declared model class and assume the declaration includes a continuous S^1 -action A and a continuous phase observable $\phi : \Sigma \rightarrow \mathbb{T}_P$ satisfying (1) on a domain D with $\deg(\chi) \neq 0$. Then the declared context belongs to COC on D . Consequently, on the correct circle target (with Haar baseline), the SFT engine posture applies: the correct output is either a WSC with explicit margins, or a finite WOB with evidence.*

Theorem 7 (Frame/gauge invariance (FEQ)). *Assume two declared contexts are related by a frame/gauge equivalence (FEQ) that preserves slice identification and conjugates the S^1 -action and phase observable on the declared domain D : there is a homeomorphism $F : D \rightarrow D'$ such that $F(A(\theta, x)) = A'(\theta, F(x))$ and $\phi' \circ F = \phi$. Then the degree identity $\deg(\phi \circ \gamma_x) = \deg(\chi)$ and the forcing/quotient dichotomy in Theorem 5 are invariant under the FEQ change of description.*

Canonical proof template: embedded loop / defect witnesses (degree and holonomy)

Theorem 8 (Embedded loop degree witness forces circle coverage). *Let Σ be a slice and let $i : S^1 \hookrightarrow \Sigma$ be a continuous embedding of a loop $E = i(S^1)$. Let $\phi : \Sigma \rightarrow \mathbb{T}_P$ be continuous and set $f := \phi \circ i : S^1 \rightarrow \mathbb{T}_P$. If $\deg(f) = k \neq 0$, then $f(S^1) = \mathbb{T}_P$. In particular, E is an embedded Syncré witness (ESW) for circle-type coverage with witness integer k .*

Proof. Identify $\mathbb{T}_P = \mathbb{R}/P\mathbb{Z}$ with the standard circle $\mathbb{T}_{2\pi}$ by linear rescaling; degree is invariant under this identification. Represent S^1 as $\mathbb{R}/2\pi\mathbb{Z}$ and lift f to a continuous $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\hat{f}(t + 2\pi) = \hat{f}(t) + 2\pi k$. By continuity, $\hat{f}([0, 2\pi])$ is an interval whose endpoints differ by $2\pi k$. Hence its image modulo 2π meets every residue class, so f is surjective onto $\mathbb{T}_{2\pi}$, equivalently onto $\mathbb{T}_P$ [1]. $\square$

Theorem 9 (Holonomy witness forces circle coverage). *Let $i : S^1 \hookrightarrow \Sigma$ and $\phi : \Sigma \rightarrow \mathbb{T}_P$ be as above. Assume there exists a lift atlas along E with local lifts F and associated connection form α such that for the loop $\gamma := i$,*

$$\int_{\gamma} \alpha = mP \quad \text{for some integer } m \neq 0.$$

Then $\phi(E) = \mathbb{T}_P$, and the induced map $f = \phi \circ i$ has nonzero degree (indeed degree m) under the standard circle identification.

Proof. On each chart in the lift atlas, the connection form is $\alpha = dF$ for a local lift F . By additivity of integrals and the cocycle compatibility on overlaps, the loop integral of α equals the total increment of any lifted phase around one circuit of γ . Thus the lifted phase changes by mP on one circuit, i.e. the winding number is $m \neq 0$. A nonzero winding number implies nonzero degree, so surjectivity follows from Theorem 8. $\square$

Theorem 10 (Robustness of embedded witnesses under LCM/HMR). *Let $\tilde{\phi} : \Sigma \rightarrow \mathbb{T}_P$ be a perturbation of ϕ and write $\tilde{f} := \tilde{\phi} \circ i$.*

- **LDW stability (lift closeness).** *Suppose f and $\tilde{f}$ admit lifts $\hat{f}, \hat{\tilde{f}} : \mathbb{R} \rightarrow \mathbb{R}$ (after identifying $\mathbb{T}_P$ with $\mathbb{T}_{2\pi}$) such that*

$$\sup_{t \in [0, 2\pi]} |\hat{f}(t) - \hat{\tilde{f}}(t)| < \pi.$$

Then $\deg(\tilde{f}) = \deg(f)$.

- **HW stability (holonomy margin).** *Suppose α and $\tilde{\alpha}$ are connection forms on E representing the declared lift/connection model for ϕ and $\tilde{\phi}$, and*

$$\left| \int_{\gamma} (\alpha - \tilde{\alpha}) \right| < P/2.$$

Then the holonomy integer multiple is unchanged.

Consequently, any ESW with nonzero witness integer remains an ESW under perturbations satisfying the corresponding margin inequality (the engine records these as $\text{LCM} < \pi$ and $\text{HMR} < P/2$ in the Synchronic Stability Modulus).

Proof. For LDW, the lift-closeness bound provides a homotopy through lifted maps that never crosses a branch cut, so winding number is constant along the homotopy and degree is preserved.

For HW, the integral bound forces $\int_{\gamma} \tilde{\alpha}$ to remain within $P/2$ of the integer multiple $\int_{\gamma} \alpha = mP$, hence the nearest integer multiple is still mP and the holonomy class is unchanged. $\square$

Canonical proof template: multi-cycle forcing and correct coupled targets

Theorem 11 ($\mathbb{T}^k$ -action forcing template: image subtorus and effective target rank). *Let Σ be a slice and let $A : \mathbb{T}^k \times \Sigma \rightarrow \Sigma$ be a continuous $\mathbb{T}^k$ -action. Let $\Phi : \Sigma \rightarrow \mathbb{T}_{\mathbf{P}}$ be continuous. Assume there exists a continuous homomorphism $\chi : \mathbb{T}^k \rightarrow \mathbb{T}_{\mathbf{P}}$ such that on a declared domain $D \subseteq \Sigma$,*

$$\Phi(A(u, x)) = \Phi(x) \oplus \chi(u) \quad \forall (u, x) \in \mathbb{T}^k \times D. \quad (2)$$

Define the (possibly lower-dimensional) response subtorus

$$\text{Im}(\chi) := \chi(\mathbb{T}^k) \subseteq \mathbb{T}_{\mathbf{P}}.$$

Then for every $x \in D$,

$$\Phi(A(\mathbb{T}^k, x)) = \Phi(x) \oplus \text{Im}(\chi).$$

In particular, the action-forcing mechanism forces coverage on the coset target $\Phi(x) \oplus \text{Im}(\chi)$. A universality claim in this setting is therefore well-posed only when its declared target agrees with this coset (or an explicitly declared coupled target contained in it).

Moreover, after choosing standard identifications $\mathbb{T}^k \cong \mathbb{R}^k / 2\pi\mathbb{Z}^k$ and $\mathbb{T}_{\mathbf{P}} \cong \prod_{j=1}^m \mathbb{R} / P_j\mathbb{Z}$, there exists an integer matrix $M \in \mathbb{Z}^{m \times k}$ such that χ lifts to the linear map $t \mapsto \text{diag}(P_1/2\pi, \dots, P_m/2\pi) Mt$ on universal covers. The effective target rank is

$$r := \text{rank}(M) = \dim(\text{Im}(\chi)),$$

so the response rank determines the target rank: the correct target cannot have dimension larger than r .

Proof. Fix $x \in D$. For any $u \in \mathbb{T}^k$, (2) gives $\Phi(A(u, x)) = \Phi(x) \oplus \chi(u)$. Thus

$$\Phi(A(\mathbb{T}^k, x)) = \{\Phi(x) \oplus \chi(u) : u \in \mathbb{T}^k\} = \Phi(x) \oplus \chi(\mathbb{T}^k) = \Phi(x) \oplus \text{Im}(\chi).$$

This is exactly the coset identity.

For the rank statement, observe that continuous homomorphisms between tori are induced by integer matrices on fundamental groups. Under the standard coordinate identifications, a homomorphism $\chi : \mathbb{T}^k \rightarrow \mathbb{T}_{\mathbf{P}}$ admits a lift to a linear map on universal covers whose matrix has integer entries up to the fixed diagonal period scaling. The image $\text{Im}(\chi)$ is a closed connected subgroup of $\mathbb{T}_{\mathbf{P}}$, hence a subtorus, and its dimension equals the rank of the inducing integer matrix M . $\square$

Theorem 12 (Coupling forces a coupled target (CST) and a coupled Haar baseline). *Let $\Phi : \Sigma \rightarrow \mathbb{T}_{\mathbf{P}}$ be a multi-cycle phase observable and suppose the declaration includes a coupling law on the same*

domain D that restricts the image to a coset of a subtorus. In the integer-matrix atlas class this has the form

$$A\Phi(x) = \Psi(s) \quad \forall x \in D \cap \Sigma_s, \quad (3)$$

for a fixed small-integer matrix A (declared or inferred) and a slice-dependent offset $\Psi(s)$. Define the coupled target on slice s by

$$\text{CST}_s := \{y \in \mathbb{T}_{\mathbf{P}} : Ay = \Psi(s)\}.$$

Then any coverage, stability, or baseline statement is meaningful only on CST_s . In particular:

- If a claim uses the full torus as its target when (3) holds, the claim is ill-posed and the correct engine output is the coupling-mismatch obstruction (WOB-06).
- The correct uniform baseline in coupled settings is Haar on CST_s (equivalently, the pushforward of Haar on the free-coordinate parameterization of CST_s).
- In the presence of a $\mathbb{T}^k$ -action equivariance (2), the forced target is the intersection of the response coset from Theorem 11 with the coupled target; in the target-correct regime these coincide because the declaration chooses CST_s as the target.

Proof. The set CST_s defined by (3) is, by construction, a coset of the subtorus $\ker(A)$. Thus it is a legitimate torus-type target equipped with its intrinsic Haar measure. A statement phrased on the full torus while the dynamics and observable are confined to CST_s changes the meaning of “coverage” and “uniformity” and therefore is not a well-posed SFT claim.

If, in addition, (2) holds, then by Theorem 11 the action-forced image on any orbit is a translate of $\text{Im}(\chi)$. When the declaration is target-correct, the target selection rule chooses the coupled target coset that actually contains the image; this is exactly the role of CST_s . $\square$

Theorem 13 (Multi-cycle forcing yields a COC embedding on the correct target). *Assume a declared context provides: (PO) a multi-cycle phase observable Φ , (WM) an admissible $\mathbb{T}^k$ -action A with equivariance (2) on a domain D , and (TR) the target rule selecting the correct coset target (the response image coset, or a coupled target CST_s when coupling constraints are present). Then the declared context belongs to COC on D . Consequently, on the correct target (with Haar or coupled Haar baseline as appropriate), the SFT engine posture applies: the correct output is either a WSC with explicit margins, or a finite WOB with evidence.*

Theorem 14 (Frame/gauge invariance (FEQ) in the multi-cycle setting). *Assume two declared contexts are related by a frame/gauge equivalence (FEQ) that preserves slice identification and conjugates the $\mathbb{T}^k$ -action and phase observable on the declared domain D : there is a homeomorphism $F : D \rightarrow D'$ such that $F(A(u, x)) = A'(u, F(x))$ and $\Phi' \circ F = \Phi$. Then the coset identity in Theorem 11 and the coupled-target correctness statement in Theorem 12 are invariant under the FEQ change of description.*

Finite obstruction classification (witness-first)

Inside COC, refusal to certify has a finite set of structural reasons. These reasons are implemented as validator predicates and canonical WOB codes, so failure modes are explicit and auditable. Examples include:

- no nondegenerate response available in an equivariant action setting (slope-trivial response in the declared forcing domain),

- no circle-type orbit or embedded loop supporting the declared witness,
- equivariance or lift/holonomy consistency defect exceeding the declared tolerance,
- coupling mismatch: a full-torus target is asserted when coupling forces a coupled coset target,
- insufficient data to identify the coupled target stably at the declared tolerance (ambiguity, insufficient excitation, resonance masking).

Robustness margins

Witnesses are packaged with explicit margins so that stability is checkable under a declared perturbation model:

- degree witnesses persist under lift-closure margins $< \pi$,
- holonomy witnesses persist under loop-integral margins $< P/2$,
- action witnesses persist under quantified equivariance defect bounds recorded in the certificate.

Multi-cycle forcing and correct targets

In multi-cycle settings, forcing is asserted on the correct target: the response-image coset determined by the induced torus homomorphism and, when present, the coupled subtorus coset CST_s determined by constraints. Baselines for statistical statements are Haar on the chosen target, or coupled Haar on CST_s .

Dynamics and convergence upgrades

SFT separates preservation from mixing. Preservation hypotheses are recorded in certificates as margins; convergence to Haar is asserted only under explicit hypotheses (nonresonant increments for unique ergodicity, or a noise-to-diffusion bridge and a verified mixing inequality for exponential convergence).

Universality Scorecard (one-page verification)

Family	Mechanism(s)	Target type(s)	Representative certificates	Representative boundary codes
Rotation & periodic forcing	AWT	circle	WSC-G-AWT-0001, WS C-ATLAS-0005, WSC-A TLAS-0035	WOB-08, WOB-09, WOB-12
Embedded loops & defects	LDW, HW	embedded	WSC-E-LDW-0001, WS C-E-HW-0002, WSC-A TLAS-0001, WSC-ATL AS-0039	WOB-02, WOB-04, WOB-05
Coupled (CST)	targets CMT	cst	WSC-C-CMT-0001, WS C-ATLAS-0002, WSC-A TLAS-0044	WOB-06, WOB-13, WOB-15

Family	Mechanism(s)	Target type(s)	Representative certificates	Representative boundary codes
Torus drives & equidistribution	AWT + RNF certificate	torus	WSC-ATLAS-0021, WS C-ATLAS-0048, CC-R NF-0001	WOB-10, WOB-06
Noise-to-diffusion mixing upgrades	MXC (+ NDB bridge)	circle/torus/cst	WSC-M-MXC-0001, WS C-ATLAS-0010, WSC-A TLAS-0051, CC-NDB-M XC-0001	WOB-07
Boundary obstructions	WOB	any	WOB-ATLAS-*	WOB-01–WOB-15

All codes and predicate mappings are part of the release artifacts (`engine/contracts/`).

The next step is to keep the target correct when multiple cycles interact: coupling changes what coverage and uniformity mean.

7 Coupled Atlas (targets and baselines)

Necessity in the presence of frame equivalence and coupling

Two correctness principles prevent hidden assumptions.

First, frame or gauge changes that preserve slice correspondence and phase meaning do not change the content of a Synchré claim. Forcing, obstruction, and robustness statements are therefore expressed on equivalence classes rather than coordinate choices.

Second, coupling changes the meaning of coverage. When constraints restrict the image to a coupled target CST_s , necessity statements refer to surjectivity and stability *on* CST_s , not on the full torus. Any statistical baseline must be coupled Haar on CST_s .

Coupling constraints restrict the image to a coupled subtorus CST_s . The correct target and baseline must be selected using CTP/CHT. Table 8 summarizes the coupling classes used in SFT.

Table 8: Coupled atlas classes: correct CST targets and coupled Haar baselines.

Coupling class	CST target	Baseline (coupled Haar)
SOL: $b \ominus a = \Psi(s)$	$\{(a, a \oplus \Psi(s))\}$	$(\eta_s)_{\#} \mu_H, \eta_s(a) = (a, a \oplus \Psi(s))$
Affine: $b \ominus (\alpha \odot a) = \Psi(s)$	$\{(a, \alpha \odot a \oplus \Psi(s))\}$	$(\eta_s)_{\#} \mu_H, \eta_s(a) = (a, \alpha \odot a \oplus \Psi(s))$
Integer-matrix: $Ay = \Psi(s)$	coset of a subtorus	$(\eta_s)_{\#} \mu_H^{(m)}$ from free-coordinate parameterization

8 Robustness, certificates, and obstructions

Observed principles must persist under perturbations. SFT packages witness claims as artifacts:

- **WSC (Witness Stability Certificate):** stores witness tier, mechanism, value, perturbation model, and explicit stability margins.
- **WOB (Witness Obstruction):** stores a named failure mode with evidence and recommended next moves.

9 Dynamics boundary (preservation vs breaking)

SFT separates witness preservation from mixing. Preservation conditions are encoded as certificate margins and are summarized below.

Property	Sufficient conditions	Certificate linkage
Degree preserved (LDW)	regular transport on loop; periodic compatibility	DMG: $\max_lift_drift < \pi$ (LCM)
Holonomy preserved (HW)	$d\omega = 0$; regular connection evolution	DMG: $\max_holonomy_drift < P/2$ (HMR)
CST target persists	coupling constraint enforced (SOL/affine/matrix)	WSC assumptions include coupling class + CTP
Mixing claim valid	NDB adopted; diffusion dominates shear (MXC)	WSC includes MXC + rate in ledger

10 Dynamics and mixing: preservation vs convergence

Convergence certificates (engine-native ergodic and diffusion upgrades)

SFT treats ergodic and diffusion results as artifact outputs.

Convergence certificate (CC)

A *Convergence Certificate* records a hypothesis bundle and a quantitative conclusion:

- **RNF certificate (ergodic).** Given rationally independent increment vector ω (nonresonant drive frequencies), time-averages converge to Haar on the correct target (full torus or CST) [4, 6, 7]. The certificate stores the target and the RNF condition used.
- **NDB+MXC certificate (diffusion).** Given an explicit noise-to-diffusion bridge (NDB) producing an advection–diffusion model and a verified mixing inequality (MXC), deviations from the baseline decay exponentially with rate constants pulled directly from [8, 10] `engine/contracts/constants_thresholds.md`.

Operationally, the engine outputs CC information inside a WSC (mechanism set to MXC) or as a universality report artifact that lists the rate line used from the thresholds ledger.

Deterministic transport can preserve witnesses (degree and, under suitable conditions, holonomy), but transport alone is not asserted to be universally mixing. Convergence to Haar is conditional and becomes theorem-level once a noise-to-diffusion bridge is adopted and mixing certificate hypotheses are satisfied [4, 6, 8].

11 Appendix A: SFT engine artifacts

Appendix: SFT Engine Artifacts (Schemas, Certificates, Obstructions, Audit)

SFT is designed to ship with explicit artifact contracts and example artifacts so claims are traceable.

Schemas

- Witness Stability Certificate schema: `artifacts/schemas/witness_stability_certificate_schema.json`
- Witness Obstruction schema: `artifacts/schemas/witness_obstruction_schema.json`
- Universality Report schema: `artifacts/schemas/universality_report_schema.json`

Engine contract

- `artifacts/contracts/engine_contract.md`

Example artifacts

- `artifacts/artifacts/WSC-G-AWT-0001.json`
- `artifacts/artifacts/WSC-E-LDW-0001.json`
- `artifacts/artifacts/WSC-C-CMT-0001.json`
- `artifacts/artifacts/WSC-M-MXC-0001.json`
- `artifacts/artifacts/WOB-E-0001.json`
- `artifacts/artifacts/WOB-G-0002.json`
- `artifacts/artifacts/UNIVERSALITY_REPORT_EXAMPLE.json`

Audit

Basic checks for WSC artifacts (required fields plus margin inequalities) are performed by:

- `artifacts/audit/audit_wsc.py`

Example:

```
python3 artifacts/audit/audit_wsc.py artifacts/artifacts/WSC-E-LDW-0001.json
```

Threshold ledger

Key inequalities and rate bounds are centralized in:

- `artifacts/contracts/constants_thresholds.md`

The paper ships with artifact contracts and example objects in the accompanying bundle:

- Contracts: `artifacts/contracts/`
- Schemas: `artifacts/schemas/`
- Example artifacts: `artifacts/artifacts/`
- Audit script: `artifacts/audit/audit_wsc.py`

12 Appendix B: worked example

A worked example pack (Earth local time plus coupled solar–lunar constraint) is provided in the bundle.

Appendix: Referee Map

Appendix: Referee Map (Claim $\rightarrow$ Hypotheses $\rightarrow$ Theorem $\rightarrow$ Artifact)

This table is intended to prevent overclaiming and to make review efficient.

Claim	Status	Support (theorem pointer)	Artifact pointer
Universality of mechanism (slice-level)	conditional proved	AWT auxiliary witness (<code>Syncre_ProofProgram.md</code>)	WSC (e.g., <code>artifacts/artifacts/WSC-G-AWT-0001.json</code>)
Arena-level SCL under slice-uniform AWT	conditional proved	Arena universality contract (<code>engine/docs/arena_universality.md</code>)	WSC plus FEQ invariance
Embedded observability (ESW)	conditional proved	ESW and obstructions (<code>engine/docs/embedded_loop_witnesses.md</code>)	WSC (e.g., <code>artifacts/artifacts/WSC-E-LDW-0001.json</code>)
Robustness under perturbations	conditional proved	LCM and HMR (<code>finish_ready/sft2/sft2_syncre_stability_modulus.tex</code>)	WSC margins; audit script
Coupled target correctness	proved	CTP and coupled theorem schema (<code>engine/docs/coupled_atlas.md</code>)	WSC-C (e.g., <code>artifacts/artifacts/WSC-C-CMT-0001.json</code>)

Claim	Status	Support pointer)	(theorem	Artifact pointer
Mixing under diffusion or noise	conditional proved	Mixing certificate (<code>engine/docs/mixing_bridge_ndb.md</code>)		WSC-M or WOB-7
Global universality theorem package	stated, theorem-backed	Capstone chain (<code>global_universality/GLOBAL_SYN_CRE_UNIVERSALITY_THEOREM.md</code>)		WSC family and WOB
Nonclaims (explicit boundaries)	stated	Referee checklist (<code>engine/docs/referee_map.md</code>)		Universality report schema

Nonclaims must be preserved in all distributed copies: SFT does not claim all systems satisfy SCL; deterministic transport is not universally mixing; full-torus Haar is not the target under coupling; diffusion or noise is adopted only under explicit modeling assumptions.

Appendix: Worked Examples

Appendix: Worked Examples (auditable examples across witness tiers)

Purpose

This appendix provides three worked examples illustrating the engine discipline across witness tiers:

- global witness (AWT)
- embedded witness (loop degree or holonomy)
- coupled-target witness (CST under SOL or affine coupling)

Each example points to a certificate or obstruction artifact and gives a minimal audit command.

Example 1: Global witness via AWT (rotating body)

- Mechanism: AWT
- Tier: WTR-G
- Artifact: `artifacts/artifacts/WSC-G-AWT-0001.json`

Audit:

```
python3 artifacts/audit/audit_wsc.py artifacts/artifacts/WSC-G-AWT-0001.json
```

Example 2: Embedded witness via LDW or HW (loop-level phase winding)

- Mechanism: LDW (or HW)
- Tier: WTR-E
- Artifact: `artifacts/artifacts/WSC-E-LDW-0001.json`

Audit:

```
python3 artifacts/audit/audit_wsc.py artifacts/artifacts/WSC-E-LDW-0001.json
```

If the setting lacks a meaningful phase observable or loop subsystem, the engine should output an obstruction, for example:

- `artifacts/artifacts/WOB-E-0001.json`

Example 3: Coupled-target witness (CST target under coupling)

- Mechanism: CMT
- Tier: WTR-C
- Artifact: `artifacts/artifacts/WSC-C-CMT-0001.json`

Audit:

```
python3 artifacts/audit/audit_wsc.py artifacts/artifacts/WSC-C-CMT-0001.json
```

Mixing boundary:

- if diffusion or noise is not adopted, mixing claims should be replaced by an obstruction (for example `artifacts/artifacts/WOB-G-0002.json`)

Diagnostic remark (targets)

- Uncoupled: baseline is Haar on $\mathbb{T}_P$
- Coupled: baseline is coupled Haar on CST (CTP and CHT)

Key inequalities and rate reminders are centralized in:

- `artifacts/contracts/constants_thresholds.md`

Capstone theorem (COC $\Rightarrow$ WSC/WOB, upgrades optional)

Theorem 15 (COC engine dichotomy). *Fix the declared hypothesis regime (including target type and perturbation model). For every context in the Cyclic Observability Class (COC), the SFT engine outputs exactly one of the following:*

- *a Witness Stability Certificate (WSC) asserting coverage on the correct target, with explicit margins and a validator predicate ledger, or*

- a *Witness Obstruction (WOB)* whose canonical code identifies the first failed validator predicate, together with evidence.

Moreover, any statistical or mixing statement is an optional upgrade: it appears only when a convergence certificate is explicitly declared and its baseline matches the correct target.

This capstone statement is supported by the theorem schema templates (action and loop forcing), the predicate-exhaustion completeness lemmas, and the target-correctness axioms.

Global Syncre Universality Theorem (Integrated)

Purpose

Synchré Form Theory (SFT) is witness-first. The Synchré Coverage Law (SCL) is the slice-wise statement that the phase field attains every phase value somewhere on each slice. This package proves that in the minimal nondegenerate regime of cyclic symmetry and equivariant phase observables, coverage is forced, stable, and admits a finite obstruction catalog.

Throughout, $\mathbb{T}_P := \mathbb{R}/P\mathbb{Z}$ denotes the phase circle of period $P > 0$.

13 Universality-from-cyclic-structure schema (reference)

The schema used throughout this integrated appendix is Theorem 4 stated in the main body. This appendix develops the detailed specialization and supporting lemmas for that schema in the cyclic-symmetry regime.

14 Core definitions

Definition 4 (Sliced arena). A *sliced arena* is a decomposition $M = \bigsqcup_{s \in \mathbb{R}} \Sigma_s$.

Definition 5 (Phase field and SCL). A *phase field* is a map $\Phi : M \rightarrow \mathbb{T}_P$. Write $\Phi_s := \Phi|_{\Sigma_s}$. The *Synchré Coverage Law (SCL)* is

$$\forall s, \quad \Phi_s(\Sigma_s) = \mathbb{T}_P.$$

Definition 6 (Equivariant circle action). Let $A : S^1 \times \Sigma \rightarrow \Sigma$ be a continuous S^1 -action on a slice Σ . A *phase observable* $\phi : \Sigma \rightarrow \mathbb{T}_P$ is *equivariant* if there exists a continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P$ such that

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) \quad (\theta \in S^1, x \in \Sigma).$$

For $x \in \Sigma$, write $\text{Stab}(x) := \{\theta \in S^1 : A(\theta, x) = x\}$.

Lemma 16 (Homomorphisms have trivial-or-surjective image). Any continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P \cong S^1$ is either trivial or surjective.

Proof. The image of a connected set under a continuous map is connected. Connected subgroups of S^1 are either $\{0\}$ or S^1 . $\square$

Definition 7 (Nonzero slope). *Nonzero slope means χ is nontrivial, equivalently surjective. Under the identifications $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}_P \cong \mathbb{R}/P\mathbb{Z}$, there exists $d \in \mathbb{Z}$ such that*

$$\chi(\theta) = \frac{P}{2\pi}d\theta \pmod{P}.$$

Nonzero slope is $d \neq 0$.

Lemma 17 (Stabilizers are forced into the kernel). *If ϕ is χ -equivariant, then for every $x \in \Sigma$,*

$$\text{Stab}(x) \subseteq \ker \chi.$$

Proof. If $\theta \in \text{Stab}(x)$, then $A(\theta, x) = x$, so equivariance gives $\phi(x) = \phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, hence $\chi(\theta) = 0$. $\square$

15 Forcing by cyclic symmetry

Theorem 16 (Circle-action forcing, minimal form). *Let (Σ, A) be a slice with a continuous S^1 -action and let $\phi : \Sigma \rightarrow \mathbb{T}_P$ be χ -equivariant. If χ has nonzero slope, then*

$$\phi(\Sigma) = \mathbb{T}_P.$$

Proof. Pick any $x \in \Sigma$. By Lemma 17, the orbit $O_x \cong S^1/\text{Stab}(x)$ is compatible with χ . By equivariance, $\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, so $\phi(O_x) = \phi(x) \oplus \chi(S^1) = \mathbb{T}_P$ since χ is surjective. As $O_x \subseteq \Sigma$, we have $\phi(\Sigma) = \mathbb{T}_P$. $\square$

16 Characterization under equivariant actions

Theorem 17 (Slice-wise coverage under an equivariant circle action). *Let (Σ, A, ϕ, χ) be as above, with ϕ χ -equivariant. Then $\phi(\Sigma) = \mathbb{T}_P$ holds if and only if either*

- χ has nonzero slope, or
- χ is trivial and the induced map $\bar{\phi} : \Sigma/S^1 \rightarrow \mathbb{T}_P$ is surjective.

In particular, in the nonzero-slope regime, coverage is automatic.

Proof. If χ has nonzero slope, Theorem 16 gives coverage. If χ is trivial, equivariance means ϕ is constant on each orbit and factors through Σ/S^1 , so coverage is equivalent to surjectivity of the factor. $\square$

17 Finite obstruction dichotomy

SFT phrases universality as a dichotomy: either a witness certificate exists (WSC), or a finite obstruction object exists (WOB). In the sense fixed in `COC_definition.tex`, this dichotomy is the engine-level universality statement on COC. Write $\text{SCL}^{\text{SFT}}(\Sigma)$ for *witnessed coverage on the correct target* in the declared model class.

18 Robustness and quantitative persistence

Use the geodesic metric on $\mathbb{T}_P$, $d_{\mathbb{T}_P}([u], [v]) = \min_{n \in \mathbb{Z}} |u - v + nP|$. Define the equivariance defect on a domain $D \subseteq \Sigma$ by

$$\text{Def}_D(A, \phi, \chi) := \sup_{\theta \in S^1, x \in D} d_{\mathbb{T}_P}(\phi(A(\theta, x)), \phi(x) \oplus \chi(\theta)).$$

Theorem 18 (Robust forcing on an orbit). *Assume χ has nonzero slope and $\text{Def}_D(A, \phi, \chi) \leq \varepsilon$. Then for any $x \in D$, the image of ϕ on the orbit segment $O_x \cap D$ is ε -dense in $\mathbb{T}_P$. Moreover, if on some orbit $O \cong S^1$ the restriction admits a lift $f(t) = \frac{dP}{2\pi}t + \psi(t)$ with $\|\psi'\|_\infty < \frac{|d|P}{2\pi}$, then $\phi(O) = \mathbb{T}_P$.*

19 Multi-cycle forcing and coupled targets

Let $\mathbb{T}_{\mathbf{P}} := \prod_{j=1}^k \mathbb{T}_{P_j}$. A $\mathbb{T}^k$ -equivariant observable $\Phi : \Sigma \rightarrow \mathbb{T}_{\mathbf{P}}$ is governed by a homomorphism $\chi : \mathbb{T}^k \rightarrow \mathbb{T}_{\mathbf{P}}$, represented (after standard identifications) by an integer matrix $D \in \mathbb{Z}^{k \times k}$. Write $r := \text{rank}(D)$.

Theorem 19 (Torus forcing, correct target form). *If Φ is χ -equivariant, then for every $x \in \Sigma$,*

$$\Phi(\mathbb{T}^k \cdot x) = \Phi(x) \oplus \text{Im}(\chi),$$

a coset of the subtorus $\text{Im}(\chi) \subseteq \mathbb{T}_{\mathbf{P}}$. If the $\mathbb{T}^k$ -action is transitive on a slice Σ_s (so Σ_s is a $\mathbb{T}^k$ -torsor), then $\Phi(\Sigma_s)$ is a single coset of $\text{Im}(\chi)$; after choosing a basepoint normalization it may be identified with $\text{Im}(\chi)$. In particular, if $r = k$ then $\text{Im}(\chi) = \mathbb{T}_{\mathbf{P}}$ and every coset is the full torus, while if $r < k$ then every orbit-image is an r -dimensional subtorus coset.

If coupling constraints restrict the image to a coupled subtorus $\text{CST}_s \subseteq \mathbb{T}_{\mathbf{P}}$, the correct target is $\Phi(\Sigma_s) = \text{CST}_s$ and the correct statistical baseline is Haar on CST_s .

20 Ergodic universality

Theorem 20 (Unique ergodicity of irrational torus translations). *If $R_\omega : \mathbb{T}^k \rightarrow \mathbb{T}^k$ is translation by $\omega \in \mathbb{R}^k$ with $1, \omega_1, \dots, \omega_k$ linearly independent over $\mathbb{Q}$, then R_ω is uniquely ergodic and time averages converge to Haar.*

21 Dynamics preservation and diffusion

If dynamics commute with the symmetry action and preserve the invariants used by the witness certificates (degree/holonomy/coupled relation), then witnesses persist in time. On tori, diffusion drives distributions exponentially fast to Haar; under standard irreducibility assumptions for advection–diffusion, convergence is exponential in total variation.

22 Minimal structure and sharp boundary

For the forcing implication, the only required ingredients are a sliced arena, a nontrivial cyclic symmetry action, and an equivariant phase observable with nonzero slope. If a slice admits no admissible cyclic symmetry (no nontrivial S^1 subgroup in the admissible automorphism class), the action-forcing witness engine cannot apply; any Synchré claim must be embedded-witness based or must add structure.

23 Capstone global theorem

Theorem 21 (Global Synchronicity universality). *Let $M = \bigsqcup_s \Sigma_s$ be a sliced arena. Suppose that on each slice:*

- *there is admissible cyclic symmetry (an S^1 or $\mathbb{T}^k$ action),*
- *there exists a phase field that is equivariant with nonzero slope, with no degree/holonomy obstruction in the adopted model class,*
- *in coupled settings, the correct target CST_s is selected and witness claims are made on that target.*

Then slice-wise coverage holds on the correct target (witness-first SCL), is robust under admissible perturbations preserving the obstruction-free regime, is preserved under admissible dynamics, converges to the correct Haar target under nonresonant drives, and converges exponentially under diffusion/noise. Failures are classified by the finite WOB obstruction catalog.

24 References

References

- [1] Hatcher, Allen. *Algebraic Topology*. Cambridge University Press, 2002.
- [2] Munkres, James R.. *Topology*. Prentice Hall, 2000.
- [3] Bredon, Glen E.. *Introduction to Compact Transformation Groups*. Academic Press, 1972.
- [4] Walters, Peter. *An Introduction to Ergodic Theory*. Springer, 1982.
- [5] Katok, Anatole and Hasselblatt, Boris. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- [6] Einsiedler, Manfred and Ward, Thomas. *Ergodic Theory with a View Towards Number Theory*. Springer, 2011.
- [7] Kuipers, L. and Niederreiter, Harald. *Uniform Distribution of Sequences*. Wiley, 1974.
- [8] Pavliotis, Grigorios A.. *Stochastic Processes and Applications*. Springer, 2014.
- [9] Stroock, Daniel W.. *An Introduction to the Analysis of Paths on a Riemannian Manifold*. American Mathematical Society, 2000.
- [10] Evans, Lawrence C.. *Partial Differential Equations*. American Mathematical Society, 2010.
- [11] Arnold, V. I.. *Mathematical Methods of Classical Mechanics*. Springer, 1989.
- [12] Strogatz, Steven H.. *Nonlinear Dynamics and Chaos*. Westview Press, 1994.
- [13] Pikovsky, Arkady, Rosenblum, Michael, and Kurths, Jürgen. *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge University Press, 2001.
- [14] Chavel, Isaac. *Riemannian Geometry: A Modern Introduction*. Cambridge University Press, 2006.

- [15] Lenstra, Arjen K., Lenstra, Hendrik W., and Lovász, László. *Factoring polynomials with rational coefficients*. Mathematische Annalen, 261(4), 515–534, 1982.
- [16] Ferguson, Helaman R. P. and Bailey, David H. *A polynomial time, numerically stable integer relation algorithm*. Technical report (PSLQ), 1999.
- [17] Meyn, Sean P. and Tweedie, Richard L. *Markov Chains and Stochastic Stability*. Cambridge University Press, 2009.
- [18] Doeblin, Wolfgang. *Exposé de la théorie des chaînes simples constantes de Markov à un nombre fini d'états*. Revue Mathématique de l'Union Interbalkanique, 2, 77–105, 1937.
- [19] Ferguson, Helaman R. P. and Bailey, David H. *A Polynomial Time, Numerically Stable Integer Relation Algorithm*. RNR Technical Report RNR-91-032, July 14, 1992.