

Global Syncré Universality: Forcing, Obstructions, Robustness, and Dynamics

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Purpose

Syncré Form Theory (SFT) is witness-first. The Syncré Coverage Law (SCL) is the slice-wise statement that the phase field attains every phase value somewhere on each slice. This note presents a compact theorem package showing that, in a minimal nondegenerate regime of cyclic symmetry and equivariant phase observables, coverage is forced, stable, and admits a finite obstruction catalog.

Throughout, $\mathbb{T}_P := \mathbb{R}/P\mathbb{Z}$ denotes the phase circle of period $P > 0$.

1 Core definitions

Definition 1 (Sliced arena). *A sliced arena is a decomposition*

$$M = \bigsqcup_{s \in \mathbb{R}} \Sigma_s.$$

Definition 2 (Phase field and SCL). *A phase field is a map $\Phi : M \rightarrow \mathbb{T}_P$. Write $\Phi_s := \Phi|_{\Sigma_s}$. The Syncré Coverage Law (SCL) is*

$$\forall s, \quad \Phi_s(\Sigma_s) = \mathbb{T}_P.$$

Definition 3 (Equivariant circle action). *Let $A : S^1 \times \Sigma \rightarrow \Sigma$ be a continuous S^1 -action on a slice Σ . A phase observable $\varphi : \Sigma \rightarrow \mathbb{T}_P$ is χ -equivariant if there exists a continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P$ such that*

$$\varphi(A(\theta, x)) = \varphi(x) \oplus \chi(\theta) \quad (\theta \in S^1, x \in \Sigma).$$

For $x \in \Sigma$, write $\text{Stab}(x) := \{\theta \in S^1 : A(\theta, x) = x\}$.

Lemma 1 (Homomorphisms have trivial-or-surjective image). *Any continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P \cong S^1$ is either trivial or surjective.*

Proof. The image of a connected set under a continuous map is connected. Connected subgroups of S^1 are either $\{0\}$ or S^1 . $\square$

Definition 4 (Nonzero slope). *Nonzero slope means χ is nontrivial, equivalently surjective. Under the identifications $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}_P \cong \mathbb{R}/P\mathbb{Z}$, there exists $d \in \mathbb{Z}$ such that*

$$\chi(\theta) = \frac{P}{2\pi} d\theta \pmod{P}.$$

Nonzero slope is $d \neq 0$.

Lemma 2 (Stabilizers lie in the kernel). *If φ is χ -equivariant, then for every $x \in \Sigma$, $\text{Stab}(x) \subseteq \ker \chi$.*

Proof. If $\theta \in \text{Stab}(x)$ then $A(\theta, x) = x$. Equivariance gives $\varphi(x) = \varphi(A(\theta, x)) = \varphi(x) \oplus \chi(\theta)$, hence $\chi(\theta) = 0$. $\square$

2 Forcing by cyclic symmetry

Theorem 1 (Circle-action forcing, minimal form). *Let (Σ, A) be a slice with a continuous S^1 -action and let $\varphi : \Sigma \rightarrow \mathbb{T}_P$ be χ -equivariant. If χ has nonzero slope, then*

$$\varphi(\Sigma) = \mathbb{T}_P.$$

Proof. Fix $x \in \Sigma$ and consider its orbit $O_x := \{A(\theta, x) : \theta \in S^1\}$. By equivariance,

$$\varphi(A(\theta, x)) = \varphi(x) \oplus \chi(\theta),$$

so $\varphi(O_x) = \varphi(x) \oplus \chi(S^1) = \mathbb{T}_P$ since χ is surjective. Hence $\varphi(\Sigma) = \mathbb{T}_P$. $\square$

3 Characterization under an equivariant action

Theorem 2 (Coverage under an equivariant circle action). *Let $(\Sigma, A, \varphi, \chi)$ be as above, with φ χ -equivariant. Then $\varphi(\Sigma) = \mathbb{T}_P$ holds if and only if either*

- χ has nonzero slope, or
- χ is trivial and the induced map $\bar{\varphi} : \Sigma/S^1 \rightarrow \mathbb{T}_P$ is surjective.

In particular, in the nonzero-slope regime, coverage is automatic.

Proof. If χ has nonzero slope, Theorem 1 gives coverage. If χ is trivial, equivariance means φ is constant on each orbit and therefore factors through the orbit space Σ/S^1 ; coverage is equivalent to surjectivity of $\bar{\varphi}$. Conversely, if χ is trivial and $\bar{\varphi}$ is surjective, then φ is surjective. $\square$

4 Witness-or-obstruction form

SFT packages universality as a dichotomy: either a witness certificate exists (WSC), or a finite obstruction object exists (WOB). Write $\text{SCL}_{\text{SFT}}(\Sigma)$ for witnessed coverage on the correct target in the declared model class.

Theorem 3 (Witness-or-obstruction dichotomy). *Fix an admissible model class. On a slice Σ , exactly one of the following occurs:*

- a finite witness certificate (WSC) exists proving correct-target coverage (full circle/torus or the selected coupled subtorus), or
- a finite obstruction object (WOB) exists from the catalog:

{orbit collapse, equivariance defect, holonomy defect, resonant locking, coupling mismatch, disconnected domain}

Equivalently,

$$\neg \text{SCL}_{\text{SFT}}(\Sigma) \iff \text{some WOB obstruction code occurs.}$$

5 Robustness (quantitative)

Use the geodesic metric on $\mathbb{T}_P$,

$$d_{\mathbb{T}_P}([u], [v]) = \min_{n \in \mathbb{Z}} |u - v + nP|.$$

Define the equivariance defect on a domain $D \subseteq \Sigma$ by

$$\text{Def}_D(A, \varphi, \chi) := \sup_{\theta \in S^1, x \in D} d_{\mathbb{T}_P}(\varphi(A(\theta, x)), \varphi(x) \oplus \chi(\theta)).$$

Theorem 4 (Robust forcing on an orbit). *Assume χ has nonzero slope and $\text{Def}_D(A, \varphi, \chi) \leq \varepsilon$. Then for any $x \in D$, the image of φ on the orbit segment $O_x \cap D$ is ε -dense in $\mathbb{T}_P$. Moreover, if on some orbit $O \cong S^1$ the restriction admits a lift*

$$f(t) = \frac{dP}{2\pi} t + \psi(t) \quad \text{with} \quad \|\psi'\|_\infty < \frac{|d|P}{2\pi},$$

then $\varphi(O) = \mathbb{T}_P$.

6 Multi-cycle forcing and coupled targets

Let $\mathbb{T}_{\mathbf{P}} := \prod_{j=1}^k \mathbb{T}_{P_j}$. A T^k -equivariant observable $\Phi : \Sigma \rightarrow \mathbb{T}_{\mathbf{P}}$ is governed by a homomorphism $\chi : T^k \rightarrow \mathbb{T}_{\mathbf{P}}$, represented (after standard identifications) by an integer matrix $D \in \mathbb{Z}^{k \times k}$. Write $r := \text{rank}(D)$.

Theorem 5 (Torus forcing, correct-target form). *If Φ is χ -equivariant, then*

$$\Phi(\Sigma) = \text{Im}(\chi).$$

In particular, if $r = k$, then $\Phi(\Sigma) = \mathbb{T}_{\mathbf{P}}$, and if $r < k$ then $\Phi(\Sigma)$ is an r -dimensional subtorus $\text{Im}(\chi)$.

If coupling constraints restrict the image to a coupled subtorus coset $\text{CST}_s \subseteq \mathbb{T}_{\mathbf{P}}$, the correct target is $\Phi(\Sigma_s) = \text{CST}_s$ and the correct statistical baseline is Haar on CST_s .

7 Ergodic universality

Theorem 6 (Unique ergodicity of irrational torus translations). *If $R_\omega : T^k \rightarrow T^k$ is translation by $\omega \in \mathbb{R}^k$ with $1, \omega_1, \dots, \omega_k$ linearly independent over $\mathbb{Q}$, then R_ω is uniquely ergodic and time averages converge to Haar.*

8 Dynamics preservation and diffusion

If dynamics commute with the symmetry action and preserve the invariants used by the witness certificates (degree, holonomy class, coupled relation), then witnesses persist in time. On tori, diffusion drives distributions exponentially fast to Haar; under standard irreducibility assumptions for advection–diffusion, convergence is exponential in total variation.

9 Minimal structure and sharp boundary

For the forcing implication, the only required ingredients are a sliced arena, a nontrivial cyclic symmetry action, and an equivariant phase observable with nonzero slope. If a slice admits no admissible cyclic symmetry (no nontrivial S^1 subgroup in the admissible automorphism class), the action-forcing engine cannot apply; any Syncré claim must be embedded-witness based or must add structure.

10 Capstone global theorem

Theorem 7 (Global Syncré universality). *Let $M = \bigsqcup_s \Sigma_s$ be a sliced arena. Suppose that on each slice:*

- *there is admissible cyclic symmetry (an S^1 or T^k action),*
- *there exists a phase field that is equivariant with nonzero slope, with no degree/holonomy obstruction in the adopted model class,*
- *in coupled settings, the correct target CST_s is selected and witness claims are made on that target.*

Then slice-wise coverage holds on the correct target (witness-first SCL), is robust under admissible perturbations preserving the obstruction-free regime, is preserved under admissible dynamics, converges to the correct Haar target under nonresonant drives, and converges exponentially under diffusion/noise. Failures are classified by the finite WOB obstruction catalog.