

Global Syncré Universality: Forcing, Obstructions, Robustness, and Dynamics

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Purpose

Syncré Form Theory (SFT) is witness-first. The Syncré Coverage Law (SCL) is the slice-wise statement that the phase field attains every phase value somewhere on each slice. This package proves that in the minimal nondegenerate regime of cyclic symmetry and equivariant phase observables, coverage is forced, stable, and admits a finite obstruction catalog.

Throughout, $\mathbb{T}_P := \mathbb{R}/P\mathbb{Z}$ denotes the phase circle of period $P > 0$.

1 Core definitions

Definition 1 (Sliced arena). *A sliced arena is a decomposition $M = \bigsqcup_{s \in \mathbb{R}} \Sigma_s$.*

Definition 2 (Phase field and SCL). *A phase field is a map $\Phi : M \rightarrow \mathbb{T}_P$. Write $\Phi_s := \Phi|_{\Sigma_s}$. The Syncré Coverage Law (SCL) is*

$$\forall s, \quad \Phi_s(\Sigma_s) = \mathbb{T}_P.$$

Definition 3 (Equivariant circle action). *Let $A : S^1 \times \Sigma \rightarrow \Sigma$ be a continuous S^1 -action on a slice Σ . A phase observable $\phi : \Sigma \rightarrow \mathbb{T}_P$ is equivariant if there exists a continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P$ such that*

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) \quad (\theta \in S^1, x \in \Sigma).$$

For $x \in \Sigma$, write $\text{Stab}(x) := \{\theta \in S^1 : A(\theta, x) = x\}$.

Lemma 1 (Homomorphisms have trivial-or-surjective image). *Any continuous homomorphism $\chi : S^1 \rightarrow \mathbb{T}_P \cong S^1$ is either trivial or surjective.*

Proof. The image of a connected set under a continuous map is connected. Connected subgroups of S^1 are either $\{0\}$ or S^1 . $\square$

Definition 4 (Nonzero slope). *Nonzero slope means χ is nontrivial, equivalently surjective. Under the identifications $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}_P \cong \mathbb{R}/P\mathbb{Z}$, there exists $d \in \mathbb{Z}$ such that*

$$\chi(\theta) = \frac{P}{2\pi} d \theta \mod P.$$

Nonzero slope is $d \neq 0$.

Lemma 2 (Stabilizers are forced into the kernel). *If ϕ is χ -equivariant, then for every $x \in \Sigma$,*

$$\text{Stab}(x) \subseteq \ker \chi.$$

Proof. If $\theta \in \text{Stab}(x)$, then $A(\theta, x) = x$, so equivariance gives $\phi(x) = \phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, hence $\chi(\theta) = 0$. $\square$

2 Pass 1: minimal forcing theorem

Theorem 1 (Circle-action forcing, minimal form). *Let (Σ, A) be a slice with a continuous S^1 -action and let $\phi : \Sigma \rightarrow \mathbb{T}_P$ be χ -equivariant. If χ has nonzero slope, then*

$$\phi(\Sigma) = \mathbb{T}_P.$$

Proof. Pick any $x \in \Sigma$. By Lemma ??, the orbit $O_x \cong S^1/\text{Stab}(x)$ is compatible with χ . By equivariance, $\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, so $\phi(O_x) = \phi(x) \oplus \chi(S^1) = \mathbb{T}_P$ since χ is surjective. As $O_x \subseteq \Sigma$, we have $\phi(\Sigma) = \mathbb{T}_P$. $\square$

3 Pass 1 upgrade: necessary and sufficient conditions

Theorem 2 (Slice-wise coverage under an equivariant circle action). *Let (Σ, A, ϕ, χ) be as above, with ϕ χ -equivariant. Then $\phi(\Sigma) = \mathbb{T}_P$ holds if and only if either*

- χ has nonzero slope, or
- χ is trivial and the induced map $\bar{\phi} : \Sigma/S^1 \rightarrow \mathbb{T}_P$ is surjective.

In particular, in the nonzero-slope regime, coverage is automatic.

Proof. If χ has nonzero slope, Theorem ?? gives coverage. If χ is trivial, equivariance means ϕ is constant on each orbit and factors through Σ/S^1 , so coverage is equivalent to surjectivity of the factor. $\square$

4 Pass 2: finite obstruction catalog (witness-first form)

SFT phrases universality as a dichotomy: either a witness certificate exists (WSC), or a finite obstruction object exists (WOB). Write $\text{SCL}^{\text{SFT}}(\Sigma)$ for *witnessed coverage on the correct target* in the declared model class.

Theorem 3 (Witness-or-obstruction dichotomy). *Fix an admissible model class. On a slice Σ , exactly one of the following occurs:*

- a finite witness certificate (WSC) exists proving the correct target coverage (full circle/torus or the selected coupled subtorus), or
- a finite obstruction object (WOB) exists from the catalog: {slope-zero, orbit-collapse, equivariance-defect, holonomy-defect, resonant-locking, coupling-mismatch, disconnected-slice, mixing-not-justified}.

Equivalently,

$$\neg \text{SCL}^{\text{SFT}}(\Sigma) \iff \text{some WOB obstruction code occurs.}$$

5 Pass 3: robustness (quantitative)

Use the geodesic metric on $\mathbb{T}_P$, $d_{\mathbb{T}_P}([u], [v]) = \min_{n \in \mathbb{Z}} |u - v + nP|$. Define the equivariance defect on a domain $D \subseteq \Sigma$ by

$$\text{Def}_D(A, \phi, \chi) := \sup_{\theta \in S^1, x \in D} d_{\mathbb{T}_P}(\phi(A(\theta, x)), \phi(x) \oplus \chi(\theta)).$$

Theorem 4 (Robust forcing on an orbit). *Assume χ has nonzero slope and $\text{Def}_D(A, \phi, \chi) \leq \varepsilon$. Then for any $x \in D$, the image of ϕ on the orbit segment $O_x \cap D$ is ε -dense in $\mathbb{T}_P$. Moreover, if on some orbit $O \cong S^1$ the restriction admits a lift $f(t) = \frac{dP}{2\pi}t + \psi(t)$ with $\|\psi'\|_\infty < \frac{|d|P}{2\pi}$, then $\phi(O) = \mathbb{T}_P$.*

6 Pass 4: multi-cycle forcing and coupled targets

Let $\mathbb{T}_{\mathbf{P}} := \prod_{j=1}^k \mathbb{T}_{P_j}$. A $\mathbb{T}^k$ -equivariant observable $\Phi : \Sigma \rightarrow \mathbb{T}_{\mathbf{P}}$ is governed by a homomorphism $\chi : \mathbb{T}^k \rightarrow \mathbb{T}_{\mathbf{P}}$, represented (after standard identifications) by an integer matrix $D \in \mathbb{Z}^{k \times k}$. Write $r := \text{rank}(D)$.

Theorem 5 (Torus forcing, correct target form). *If Φ is χ -equivariant, then*

$$\Phi(\Sigma) = \text{Im}(\chi).$$

In particular, if $r = k$, then $\Phi(\Sigma) = \mathbb{T}_{\mathbf{P}}$, and if $r < k$ then $\Phi(\Sigma)$ is the r -dimensional subtorus $\text{Im}(\chi)$.

If coupling constraints restrict the image to a coupled subtorus $\text{CST}_s \subseteq \mathbb{T}_{\mathbf{P}}$, the correct target is $\Phi(\Sigma_s) = \text{CST}_s$ and the correct statistical baseline is Haar on CST_s .

7 Pass 5: ergodic universality

Theorem 6 (Unique ergodicity of irrational torus translations). *If $R_\omega : \mathbb{T}^k \rightarrow \mathbb{T}^k$ is translation by $\omega \in \mathbb{R}^k$ with $1, \omega_1, \dots, \omega_k$ linearly independent over $\mathbb{Q}$, then R_ω is uniquely ergodic and time averages converge to Haar.*

8 Pass 6–7: dynamics preservation and diffusion

If dynamics commute with the symmetry action and preserve the invariants used by the witness certificates (degree/holonomy/coupled relation), then witnesses persist in time. On tori, diffusion drives distributions exponentially fast to Haar; under standard irreducibility assumptions for advection–diffusion, convergence is exponential in total variation.

9 Pass 8–9: minimal structure and sharp boundary

For the forcing implication, the only required ingredients are a sliced arena, a nontrivial cyclic symmetry action, and an equivariant phase observable with nonzero slope. If a slice admits no admissible cyclic symmetry (no nontrivial S^1 subgroup in the admissible automorphism class), the action-forcing witness engine cannot apply; any Synchré claim must be embedded-witness based or must add structure.

10 Pass 10: capstone global theorem

Theorem 7 (Global Synchré universality). *Let $M = \bigsqcup_s \Sigma_s$ be a sliced arena. Suppose that on each slice:*

- *there is admissible cyclic symmetry (an S^1 or $\mathbb{T}^k$ action),*

- *there exists a phase field that is equivariant with nonzero slope, with no degree/holonomy obstruction in the adopted model class,*
- *in coupled settings, the correct target CST_s is selected and witness claims are made on that target.*

Then slice-wise coverage holds on the correct target (witness-first SCL), is robust under admissible perturbations preserving the obstruction-free regime, is preserved under admissible dynamics, converges to the correct Haar target under nonresonant drives, and converges exponentially under diffusion/noise. Failures are classified by the finite WOB obstruction catalog.