Global Syncré Universality: Forcing, Obstructions, Robustness, and Dynamics

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Purpose

Syncré Form Theory (SFT) is witness-first. The Syncré Coverage Law (SCL) is the slice-wise statement that the phase field attains every phase value somewhere on each slice. This note proves that in the minimal nondegenerate regime of cyclic symmetry and equivariant phase observables, coverage is forced, stable, and admits a finite obstruction catalog. Throughout, $\mathbb{T}_P := \mathbb{R}/P\mathbb{Z}$ denotes the phase circle of period $P > 0$.

Definition 1 (Cyclic Observability Class (COC))**.** *The* Cyclic Observability Class *(*COC*) is the class of* declared *SFT contexts* c *satisfying all of the following three conditions (and it is maximal by definition, since membership is exactly these conditions):*

- *a* phase observable *is declared on the relevant slice/subsystem: a cycle-valued map (circle/torus/coupled target) with the minimal regularity needed to define the intended winding/holonomy notion and to test nontriviality;*
- *an* admissible witness mechanism *is declared (symmetry action, loop/holonomy, embedded subsystem witness, defect loop, or another explicitly listed mechanism), so the engine has a prescribed route to either a WSC or to a named obstruction code;*
- *a* correct target rule *is satisfied: the target type (circle/torus/CST) and any baseline statements are defined on the correct target (Haar on the target, coupled Haar on CST when coupling is present).*

Definition 2 (Universality on a class of contexts)**.** *Fix a class of contexts* $\mathcal{C}$*. The SFT engine is* universal on $\mathcal{C}$ *if for every declared context* $\mathsf{c} \in \mathcal{C}$ *and every slice/subsystem where the claim is posed, the engine returns exactly one of:*

- *a* Witness Stability Certificate *(WSC) proving coverage on the correct target in* c *(with explicit margins), or*
- *a* Witness Obstruction *(WOB) belonging to a finite obstruction catalog (with evidence), certifying that the claim is blocked in* c*.*

Global universality means: universal on COC**.** In SFT, a statement labeled *global universality* is a universality statement on COC: for every context in COC, the engine returns a WSC or a WOB on the correct target; outside COC, claims are either inapplicable (target/predicates not well-posed) or produce a boundary-type obstruction rather than a forced-coverage theorem.

COC checklist (entrywise). A declared context is treated as belonging to COConly when all of the following are explicitly provided:

- **Phase observable (PO).** A named cycle-valued observable on the intended slice/subsystem (circle/torus/CST), together with the minimum regularity assumptions needed to define its winding/holonomy notion and to test nontriviality.
- **Admissible witness mechanism (WM).** A declared mechanism from the admissible menu (symmetry action, loop/holonomy, embedded subsystem witness, defect loop, or another explicitly listed mechanism) that specifies what the engine must attempt to certify.
- **Target rule (TR).** A declared target type (circle/torus/CST) and baseline convention on that target (Haar on the target; coupled Haar on CST when coupling is present).

When any checklist item fails, the correct engine output is a named obstruction (WOB) rather than a coverage certificate.

Well-posedness spine: what a universality claim means in SFT

SFT treats universality as a statement with an explicit boundary. The boundary is not a philosophical warning; it is part of the formal interface. Every claim is posed in a *declared context* and every answer is a finite object. This is the only sense in which SFT uses the word *meaning*: a claim is meaningful exactly when it has an auditable certificate-or-obstruction output.

Definition 3 (Declared SFT claim)**.** *A declared SFT claim is a tuple*

$$\mathsf{q} = (\mathsf{c},\ \mathsf{T},\ \mathsf{S}),$$

where c *is a declared context (with a phase observable and an admissible witness mechanism),* T *is the declared* correct target *(circle/torus/coupled target) together with its baseline convention (Haar on the target; coupled Haar on CST when coupling is present), and* S *is the concrete Syncré statement being tested (coverage, stability, and any declared upgrade).*

> **Well-posedness axiom (WSC/WOB meaning).** For every declared claim q, the SFT engine returns exactly one of the following two outputs.
>
> - **WSC (Witness Stability Certificate).** A finite certificate proving the stated claim on the declared correct target, together with explicit margins that make the claim stable in the declared perturbation model.
> - **WOB (Witness Obstruction).** A finite obstruction report from a named catalog, with evidence, asserting that certification is blocked or that the claim is inapplicable in the declared regime.
>
> The claim is *meaningful in SFT* if and only if it is posed as such an engine query and therefore admits a WSC/WOB output.

The sharp boundary can be read in one glance.

Where the claim lives	**Correct output**
Inside COC (phase observable + witness route + target rule)	WSC or a finite WOB (structural failure mode)
Outside COC (missing phase / missing witness route / wrong target rule)	Boundary-type WOB (inapplicable or ill-posed)

How to read any "universality" sentence in this project.

- First check whether the context is declared as belonging to COC (the checklist is part of the definition).
- Then check the target: circle, full torus, or a coupled target CST.
- Then check the object-level output: a WSC (with margins) or a WOB (with a named obstruction code and evidence).

This spine is the project's scope wall in mathematical form. It prevents universality from becoming vague: every use of the word is anchored to the named class (COC) defined by the interface conditions and to an explicit certificate-or-obstruction output.

Target correctness axiom block (CTP/CHT). SFT treats *target correctness* as part of the meaning of any certificate-or-obstruction claim. A declared claim must state (i) the *target type* (circle / full torus / coupled target) and (ii) the *baseline measure* used for any comparison, stability, or convergence upgrade.

- **Circle or full torus targets.** When the correct target is a compact group target $\mathbb{T}_P$ or $\mathbb{T}_{\mathbf{P}}$, the baseline is the (unique) Haar probability measure on that target.
- **Coupled targets (CST).** When coupling constraints restrict the image to a coupled subtorus coset,

  $$\mathrm{CST}_s \ := \ \{y \in \mathbb{T}_{\mathbf{P}} : Ay = \Psi(s)\}, \qquad A \in \mathbb{Z}^{r\times d},$$

  then the correct target is CST_s (not the ambient torus) and the baseline is the (unique) translation-invariant probability measure on CST_s. Equivalently, for any parameterization $\eta_s : \mathbb{T}^m_{P_*} \to \mathrm{CST}_s$, the baseline is the pushforward of Haar:

  $$\mu^{\mathrm{CST}}_{\mathrm{H},s} := (\eta_s)_{\#}\mu^{(m)}_{\mathrm{H}}.$$

CTP (Coupled Target Principle). If the declared context includes coupling constraints yielding a coupled target CST_s, then all coverage statements, stability margins, and upgrades are to be interpreted *relative to* CST_s. Using the ambient torus as the target in this regime is not a weaker claim; it is a *different* claim.

CHT (Coupled Haar Theorem, as baseline convention). In the coupled regime, the baseline for comparisons is $\mu^{\mathrm{CST}}_{\mathrm{H},s}$. Any convergence or mixing claim must specify this baseline when the target is CST_s.

Corollary (wrong target/baseline forces a boundary obstruction). If a declared claim uses a target/baseline pair incompatible with the declared coupling structure (e.g., it uses full-torus Haar when the correct target is CST_s), then the claim is *ill-posed in SFT*. The correct engine output in that case is the coupling-mismatch obstruction `WOB-06-coupling-mismatch`.

Target correctness decision flow

This page is the canonical interface for selecting the *correct target* and *baseline* in SFT. It is written to match the WSC schema fields `target_type` and `baseline`. The point is simple: if the target is chosen correctly, the engine can attempt certification; if the target is chosen incorrectly, the correct output is a named obstruction.

Decision	Action (what you declare)	If violated, output
A. Is the claim in COC**?**	Declare a phase observable and an admissible witness route (rotation / loop-degree / holonomy / multi-cycle) and state the target rule you are using.	If any ingredient is missing, the claim is outside COC: boundary-type obstruction (typically **WOB-01**).
B. Is coupling present?	If the model includes coupling constraints across phases (an integer constraint matrix χ or an equivalent coupling law), set `target_type=CST` and declare `baseline=coupledHaaronCST`.	Using circle/torus baseline under coupling, or failing to compute/declare CST: **WOB-06**.
C. Single-cycle circle observable?	If the witness route is a circle action or a single loop/defect witness producing an integer $k \in \mathbb{Z}$, set `target_type=circle` (or `embedded_circle` for subsystem-only claims) and `baseline=Haar on` $\mathbb{T}_P$.	If every admissible witness integer is 0: **WOB-02**. If no loop/orbit exists in the declared domain: **WOB-01**.
D. Multi-cycle torus observable?	If the witness route is a $\mathbb{T}^k$ action with response map induced by an integer matrix χ, compute the effective rank $r := \mathrm{rank}(\chi)$. • If $r = k$, set `target_type=torus` and `baseline=Haar on` $\mathbb{T}^k$. • If $r < k$, set `target_type=subtorus_rank_r` and `baseline=Haar on the induced` r`-torus (the` χ`-image)`.	Claiming full-torus forcing when the response rank is deficient (or rational locking collapses the sweep): **WOB-10**.
E. Target-aware upgrades?	Any convergence or mixing upgrade must explicitly name the baseline it converges to. If the upgrade requires diffusion/mixing assumptions, record them and point to the declared evidence artifact.	Mixing/convergence claimed without an admissible hypothesis package: **WOB-07**.

Allowed upgrades (summary).

Target	Baseline	Upgrades allowed (when justified)
Circle / embedded circle	Haar on $\mathbb{T}_P$	Stability margins (LCM/HMR), anti-collapse bounds, and convergence-to-Haar if a mixing model is declared.
Full torus $\mathbb{T}^k$	Haar on $\mathbb{T}^k$	Stability margins; rank-aware coverage; convergence-to-Haar if a mixing model is declared.
Subtorus rank r	Haar on induced r-torus	Same upgrades, but always relative to the induced target (never the ambient torus unless $r = k$).
CST (coupled target)	Coupled Haar on CST	All upgrades, but always relative to coupled Haar; any non-coupled baseline is ill-posed (CTP/CHT).

1 Universality-from-cyclic-structure schema

Theorem 1 (Universality-from-cyclic-structure (COC embedding schema))**.** *Fix a declared model class and let Σ denote the slice/subsystem on which a Syncré claim is posed. Assume the declaration provides:*

- ***Phase observable (PO).*** *A cycle-valued observable $\phi : \Sigma \to \mathbb{T}_P$ or $\Phi : \Sigma \to \mathbb{T}_{\mathbf{P}}$ with the minimal regularity needed to define the intended winding/holonomy notion and to test nontriviality.*

- ***Witness mechanism (WM).*** *At least one admissible mechanism from the SFT menu, in one of the following structural forms:*
  - Action forcing. *An admissible continuous action $A : G \times D \to D$ on a domain $D \subseteq \Sigma$ with $G = S^1$ or $G = \mathbb{T}^k$, together with a continuous homomorphism $\chi : G \to H$ to the declared target group H such that ϕ or Φ is χ-equivariant on D in the declared tolerance (exactly or with a stated defect bound). The action-forcing mechanism is* nondegenerate *when the induced map on the declared target has nontrivial cyclic response (nonzero slope in the S^1 case; positive rank on the target coordinates in the $\mathbb{T}^k$ case).*
  - Embedded loop/holonomy. *An embedded loop $\gamma : S^1 \to \Sigma$ (or loop in an embedded subsystem) together with either a nonzero degree witness* $\deg(\phi \circ \gamma) \neq 0$ *or a nonzero holonomy multiple in a declared lift/connection model.*

- ***Target rule (TR).*** *A declared target type (circle/torus/coupled subtorus) and baseline convention on that target (Haar on the target, coupled Haar on CST when coupling is present).*

Then the declared context belongs to COC*. Consequently, the SFT universality posture applies on this context: on the declared target, the correct engine output is either a Witness Stability Certificate (WSC) with explicit margins, or a finite Witness Obstruction (WOB) from the obstruction catalog with evidence.*

Canonical cyclic families covered by the schema

The theorem is a *schema*: it reduces broad families to the same three-checklist declaration (PO/WM/TR).

The circle-action / periodic-forcing family admits a fully formal proof template in Theorem 3. Each line below is a structural criterion (not a physical promise): if the criterion holds in the adopted model class, it produces a COC embedding.

- **Periodic forcing / time-shift symmetry.** If the declared subsystem admits a continuous periodic parameter (identified with S^1) acting by reparameterization on states, and the phase observable is equivariant under that shift with nonzero slope on the declared phase circle, then the action-forcing mechanism applies.
- **Rotation / rigid cyclic response.** If the declared subsystem carries an admissible S^1-action (or a torus action in the multi-cycle setting) and the observable is equivariant with nontrivial induced homomorphism on the declared target, then forcing applies on the corresponding orbit-image (full circle when slope is nonzero; subtorus/coset when rank is deficient and the target rule selects CST).
- **Wave-train / translation-phase reduction.** If a periodic direction induces an admissible S^1-translation action on a domain and the phase observable is equivariant under translation with nonzero slope, the situation is again an action-forcing context.
- **Limit-cycle / periodic-orbit phase.** If the declared dynamics exhibits a periodic orbit equipped with a continuous phase coordinate (so that motion along the orbit defines an S^1-action on that orbit) and the observable responds with nonzero winding along that cycle, then either action-forcing or an embedded loop degree witness certifies cyclic observability.
- **Defect loop / winding witness.** If the declaration provides a loop around a distinguished locus (a boundary component, defect core, or excluded set) on which the phase observable has nonzero winding (or nonzero holonomy multiple in a lift model), then the embedded witness mechanism applies.
- **Coupled drives / multi-cycle response.** If multiple cyclic inputs generate an admissible $\mathbb{T}^k$-action and the observable is equivariant with integer-matrix response χ, then the correct target is the image subtorus (or the inferred coupled subtorus CST), and the forcing/obstruction dichotomy is posed on that target.

Engine completeness (one-page inevitability)

The global universality claim in SFT is not a slogan. It is an engine statement:

- inside COC, the engine must return either a *Witness Stability Certificate* (WSC) or a *Witness Obstruction* (WOB),
- and there is no third outcome.

The reason this feels final is that the engine is gated by a finite predicate suite.

Definition 4 (Predicate gate)**.** *Fix the canonical ordered list of validator predicates*

$$V_{\mathrm{PO}}, V_{\mathrm{WM}}, V_{\mathrm{CT}}, V_{\mathrm{ROB}}, \dots$$

as specified in `VALIDATOR_PREDICATE_SPEC.tex` *and the machine list* `engine/contracts/validator_predicates.j` *A declared context* `c` *passes the gate if every predicate required for its declared target type and mechanism passes.*

Theorem 2 (Completeness of the obstruction catalog)**.** *Fix the declared-regime contract (target type, perturbation model, and admissible mechanisms). For any declared context* $\mathsf{c} \in \mathrm{COC}$*, exactly one of the following holds:*

- ***All required predicates pass.*** *Then a WSC is admissible and the engine is permitted to certify coverage on the correct target, with explicit margins and a predicate ledger.*
- ***Some required predicate fails.*** *Let* V_{XX} *be the* first *failed predicate in the canonical order. Then the correct output is a WOB with obstruction code* `WOB-XX-...`*, and this code lies in the finite canonical codebook.*

*In particular, the WOB catalog is complete for in-regime failures: every in-*COC*refusal to certify has a canonical code, and every canonical code corresponds to a predicate failure.*

Pointers to the full chain. The proof is the combination of:

- predicate exhaustion logic: `PREDICATE_EXHAUSTION_LEMMAS.tex`,
- the canonical catalog: `WOB_CATALOG_COMPLETENESS.tex`,
- the predicate specification: `VALIDATOR_PREDICATE_SPEC.tex`,
- the capstone dichotomy theorem: `CAPSTONE_COC_ENGINE.tex`.

Canonical proof template: circle-action forcing (rotation / periodic forcing)

Theorem 3 (Circle-action forcing template)**.** *Let* Σ *be a slice and let* $A : S^1 \times \Sigma \to \Sigma$ *be a continuous* S^1*-action. Let* $\phi : \Sigma \to \mathbb{T}_P$ *be continuous. Assume there exists a continuous homomorphism* $\chi : S^1 \to \mathbb{T}_P$ *such that on a declared domain* $D \subseteq \Sigma$*,*

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) \qquad \forall (\theta, x) \in S^1 \times D. \tag{1}$$

Write $\deg(\chi) \in \mathbb{Z}$ *for the integer degree of* χ*.*

- *If* $\deg(\chi) \neq 0$*, then for every* $x \in D$ *the orbit loop* $\gamma_x : S^1 \to \Sigma$*,* $\gamma_x(\theta) = A(\theta, x)$ *satisfies*

$$\deg(\phi \circ \gamma_x) = \deg(\chi) \neq 0,$$

  and $\phi(D) = \mathbb{T}_P$*. In particular, the declaration* (Σ, ϕ) *satisfies the COC checklist on* D *with: (PO) phase observable* ϕ*, (WM) action forcing and embedded loop witness, (TR) circle target with Haar baseline.*

- *If* $\deg(\chi) = 0$*, then* ϕ *is* S^1*-invariant on* D *and factors through the orbit quotient* $q : D \to D/S^1$*: there exists* $\bar{\phi} : D/S^1 \to \mathbb{T}_P$ *with* $\phi|_D = \bar{\phi} \circ q$*. In this case, coverage on* $\mathbb{T}_P$ *can occur only through surjectivity of* $\bar{\phi}$*.*

Proof. Every continuous homomorphism $\chi : S^1 \to S^1$ has the form [3, 1]: $\chi(e^{2\pi i t}) = e^{2\pi i m t}$ for a unique $m \in \mathbb{Z}$, and $\deg(\chi) = m$.

Fix $x \in D$. Define $\gamma_x(\theta) = A(\theta, x)$. By (1),

$$(\phi \circ \gamma_x)(\theta) = \phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta).$$

Translation by the constant $\phi(x)$ does not change degree [1], hence $\deg(\phi \circ \gamma_x) = \deg(\chi)$.

If $\deg(\chi) \neq 0$, then χ is surjective onto $\mathbb{T}_P$. For any $y \in \mathbb{T}_P$, choose $\theta \in S^1$ with $\chi(\theta) = y \ominus \phi(x)$. Then $\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) = y$, so $\phi(D) = \mathbb{T}_P$. The nonzero degree identity above supplies an embedded loop/holonomy witness, and the equivariance identity supplies an admissible action-forcing witness mechanism.

If $\deg(\chi) = 0$, then χ is trivial, so (1) gives $\phi(A(\theta, x)) = \phi(x)$ for all θ and all $x \in D$. Thus ϕ is constant along orbits in D and therefore factors through the orbit space [3]: there exists $\bar{\phi}$ with $\phi|_D = \bar{\phi} \circ q$. □

Theorem 4 (Periodic forcing / rotation model as a COC embedding)**.** *Let Σ be a slice in a declared model class and assume the declaration includes a continuous S^1-action A and a continuous phase observable $\phi : \Sigma \to \mathbb{T}_P$ satisfying* (1) *on a domain D with* $\deg(\chi) \neq 0$*. Then the declared context belongs to* COC *on D. Consequently, on the correct circle target (with Haar baseline), the SFT engine posture applies: the correct output is either a WSC with explicit margins, or a finite WOB with evidence.*

Theorem 5 (Frame/gauge invariance (FEQ))**.** *Assume two declared contexts are related by a frame/gauge equivalence (FEQ) that preserves slice identification and conjugates the S^1-action and phase observable on the declared domain D: there is a homeomorphism $F : D \to D'$ such that $F(A(\theta, x)) = A'(\theta, F(x))$ and $\phi' \circ F = \phi$. Then the degree identity* $\deg(\phi \circ \gamma_x) = \deg(\chi)$ *and the forcing/quotient dichotomy in Theorem 3 are invariant under the FEQ change of description.*

Canonical proof template: embedded loop / defect witnesses (degree and holonomy)

Theorem 6 (Embedded loop degree witness forces circle coverage)**.** *Let Σ be a slice and let $i : S^1 \hookrightarrow \Sigma$ be a continuous embedding of a loop $E = i(S^1)$. Let $\phi : \Sigma \to \mathbb{T}_P$ be continuous and set $f := \phi \circ i : S^1 \to \mathbb{T}_P$. If* $\deg(f) = k \neq 0$*, then $f(S^1) = \mathbb{T}_P$. In particular, E is an embedded Syncré witness (*ESW*) for circle-type coverage with witness integer k.*

Proof. Identify $\mathbb{T}_P = \mathbb{R}/P\mathbb{Z}$ with the standard circle $\mathbb{T}_{2\pi}$ by linear rescaling; degree is invariant under this identification. Represent S^1 as $\mathbb{R}/2\pi\mathbb{Z}$ and lift f to a continuous $\widehat{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\widehat{f}(t + 2\pi) = \widehat{f}(t) + 2\pi k$. By continuity, $\widehat{f}([0, 2\pi])$ is an interval whose endpoints differ by $2\pi k$. Hence its image modulo 2π meets every residue class, so f is surjective onto $\mathbb{T}_{2\pi}$, equivalently onto $\mathbb{T}_P$ [1]. □

Theorem 7 (Holonomy witness forces circle coverage)**.** *Let $i : S^1 \hookrightarrow \Sigma$ and $\phi : \Sigma \to \mathbb{T}_P$ be as above. Assume there exists a lift atlas along E with local lifts F and associated connection form α such that for the loop $\gamma := i$,*

$$\int_\gamma \alpha = mP \qquad \text{for some integer } m \neq 0.$$

Then $\phi(E) = \mathbb{T}_P$, and the induced map $f = \phi \circ i$ has nonzero degree (indeed degree m) under the standard circle identification.

Proof. On each chart in the lift atlas, the connection form is $\alpha = dF$ for a local lift F. By additivity of integrals and the cocycle compatibility on overlaps, the loop integral of α equals the total increment of any lifted phase around one circuit of γ. Thus the lifted phase changes by mP on one circuit, i.e. the winding number is $m \neq 0$. A nonzero winding number implies nonzero degree, so surjectivity follows from Theorem 6. □

Theorem 8 (Robustness of embedded witnesses under LCM/HMR)**.** *Let $\tilde{\phi} : \Sigma \to \mathbb{T}_P$ be a perturbation of ϕ and write $\tilde{f} := \tilde{\phi} \circ i$.*

- ***LDW stability (lift closeness).*** *Suppose f and $\tilde{f}$ admit lifts $\widehat{f}, \widehat{\tilde{f}} : \mathbb{R} \to \mathbb{R}$ (after identifying $\mathbb{T}_P$ with $\mathbb{T}_{2\pi}$) such that*
$$\sup_{t \in [0,2\pi]} |\widehat{f}(t) - \widehat{\tilde{f}}(t)| < \pi.$$
Then $\deg(\tilde{f}) = \deg(f)$.

- ***HW stability (holonomy margin).*** *Suppose α and $\tilde{\alpha}$ are connection forms on E representing the declared lift/connection model for ϕ and $\tilde{\phi}$, and*
$$\left| \int_\gamma (\alpha - \tilde{\alpha}) \right| < P/2.$$
Then the holonomy integer multiple is unchanged.

Consequently, any ESW *with nonzero witness integer remains an* ESW *under perturbations satisfying the corresponding margin inequality (the engine records these as* LCM $< \pi$ *and* HMR $< P/2$ *in the Syncré Stability Modulus).*

Proof. For LDW, the lift-closeness bound provides a homotopy through lifted maps that never crosses a branch cut, so winding number is constant along the homotopy and degree is preserved.

For HW, the integral bound forces $\int_\gamma \tilde{\alpha}$ to remain within $P/2$ of the integer multiple $\int_\gamma \alpha = mP$, hence the nearest integer multiple is still mP and the holonomy class is unchanged. □

Canonical proof template: multi-cycle forcing and correct coupled targets

Theorem 9 ($\mathbb{T}^k$-action forcing template: image subtorus and effective target rank)**.** *Let Σ be a slice and let $A : \mathbb{T}^k \times \Sigma \to \Sigma$ be a continuous $\mathbb{T}^k$-action. Let $\Phi : \Sigma \to \mathbb{T}_{\mathbf{P}}$ be continuous. Assume there exists a continuous homomorphism $\chi : \mathbb{T}^k \to \mathbb{T}_{\mathbf{P}}$ such that on a declared domain $D \subseteq \Sigma$,*

$$\Phi(A(u,x)) = \Phi(x) \oplus \chi(u) \qquad \forall (u,x) \in \mathbb{T}^k \times D. \tag{2}$$

Define the (possibly lower-dimensional) response subtorus

$$\mathrm{Im}(\chi) := \chi(\mathbb{T}^k) \subseteq \mathbb{T}_{\mathbf{P}}.$$

Then for every $x \in D$,

$$\Phi\big(A(\mathbb{T}^k, x)\big) = \Phi(x) \oplus \mathrm{Im}(\chi).$$

In particular, the action-forcing mechanism forces coverage on the coset target *$\Phi(x) \oplus \mathrm{Im}(\chi)$. A universality claim in this setting is therefore well-posed only when its declared target agrees with this coset (or an explicitly declared coupled target contained in it).*

Moreover, after choosing standard identifications $\mathbb{T}^k \cong \mathbb{R}^k/2\pi\mathbb{Z}^k$ and $\mathbb{T}_{\mathbf{P}} \cong \prod_{j=1}^m \mathbb{R}/P_j\mathbb{Z}$, there exists an integer matrix $M \in \mathbb{Z}^{m\times k}$ such that χ lifts to the linear map $t \mapsto \mathrm{diag}(P_1/2\pi, \ldots, P_m/2\pi)\, Mt$ on universal covers. The effective target rank *is*

$$r := \mathrm{rank}(M) = \dim\big(\mathrm{Im}(\chi)\big),$$

so the response rank determines the target rank: the correct target cannot have dimension larger than r.

Proof. Fix $x \in D$. For any $u \in \mathbb{T}^k$, (2) gives $\Phi(A(u,x)) = \Phi(x) \oplus \chi(u)$. Thus

$$\Phi\big(A(\mathbb{T}^k, x)\big) = \{\Phi(x) \oplus \chi(u) : u \in \mathbb{T}^k\} = \Phi(x) \oplus \chi(\mathbb{T}^k) = \Phi(x) \oplus \mathrm{Im}(\chi).$$

This is exactly the coset identity.

For the rank statement, observe that continuous homomorphisms between tori are induced by integer matrices on fundamental groups. Under the standard coordinate identifications, a homomorphism $\chi : \mathbb{T}^k \to \mathbb{T}_{\mathbf{P}}$ admits a lift to a linear map on universal covers whose matrix has integer entries up to the fixed diagonal period scaling. The image $\mathrm{Im}(\chi)$ is a closed connected subgroup of $\mathbb{T}_{\mathbf{P}}$, hence a subtorus, and its dimension equals the rank of the inducing integer matrix M. $\square$

Theorem 10 (Coupling forces a coupled target (CST) and a coupled Haar baseline)**.** *Let $\Phi : \Sigma \to \mathbb{T}_{\mathbf{P}}$ be a multi-cycle phase observable and suppose the declaration includes a coupling law on the same domain D that restricts the image to a coset of a subtorus. In the integer-matrix atlas class this has the form*

$$A\,\Phi(x) = \Psi(s) \qquad \forall x \in D \cap \Sigma_s, \tag{3}$$

for a fixed small-integer matrix A (declared or inferred) and a slice-dependent offset $\Psi(s)$. Define the coupled target on slice s by

$$\mathrm{CST}_s := \{y \in \mathbb{T}_{\mathbf{P}} : Ay = \Psi(s)\}.$$

Then any coverage, stability, or baseline statement is meaningful only on CST_s. In particular:

- *If a claim uses the full torus as its target when* (3) *holds, the claim is ill-posed and the correct engine output is the coupling-mismatch obstruction (WOB-06).*
- *The correct uniform baseline in coupled settings is Haar on CST_s (equivalently, the pushforward of Haar on the free-coordinate parameterization of CST_s).*
- *In the presence of a $\mathbb{T}^k$-action equivariance* (2)*, the forced target is the intersection of the response coset from Theorem 9 with the coupled target; in the target-correct regime these coincide because the declaration chooses CST_s as the target.*

Proof. The set CST_s defined by (3) is, by construction, a coset of the subtorus $\ker(A)$. Thus it is a legitimate torus-type target equipped with its intrinsic Haar measure. A statement phrased on the full torus while the dynamics and observable are confined to CST_s changes the meaning of "coverage" and "uniformity" and therefore is not a well-posed SFT claim.

If, in addition, (2) holds, then by Theorem 9 the action-forced image on any orbit is a translate of $\mathrm{Im}(\chi)$. When the declaration is target-correct, the target selection rule chooses the coupled target coset that actually contains the image; this is exactly the role of CST_s. $\square$

Theorem 11 (Multi-cycle forcing yields a COC embedding on the correct target)**.** *Assume a declared context provides: (PO) a multi-cycle phase observable Φ, (WM) an admissible $\mathbb{T}^k$-action A with equivariance* (2) *on a domain D, and (TR) the target rule selecting the correct coset target (the response image coset, or a coupled target CST_s when coupling constraints are present). Then the declared context belongs to* COC *on D. Consequently, on the correct target (with Haar or coupled Haar baseline as appropriate), the SFT engine posture applies: the correct output is either a WSC with explicit margins, or a finite WOB with evidence.*

Theorem 12 (Frame/gauge invariance (FEQ) in the multi-cycle setting)**.** *Assume two declared contexts are related by a frame/gauge equivalence (FEQ) that preserves slice identification and conjugates the $\mathbb{T}^k$-action and phase observable on the declared domain D: there is a homeomorphism $F : D \to D'$ such that $F(A(u,x)) = A'(u, F(x))$ and $\Phi' \circ F = \Phi$. Then the coset identity in Theorem 9 and the coupled-target correctness statement in Theorem 10 are invariant under the FEQ change of description.*

2 Core definitions

Definition 5 (Sliced arena)**.** *A sliced arena is a decomposition $M = \bigsqcup_{s\in\mathbb{R}} \Sigma_s$.*

Definition 6 (Phase field and SCL)**.** *A phase field is a map $\Phi : M \to \mathbb{T}_P$. Write $\Phi_s := \Phi|_{\Sigma_s}$. The Syncré Coverage Law (SCL) is*

$$\forall s, \quad \Phi_s(\Sigma_s) = \mathbb{T}_P.$$

Definition 7 (Equivariant circle action)**.** *Let $A : S^1 \times \Sigma \to \Sigma$ be a continuous S^1-action on a slice Σ. A phase observable $\phi : \Sigma \to \mathbb{T}_P$ is* equivariant *if there exists a continuous homomorphism $\chi : S^1 \to \mathbb{T}_P$ such that*

$$\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta) \qquad (\theta \in S^1,\ x \in \Sigma).$$

For $x \in \Sigma$, write $\mathrm{Stab}(x) := \{\theta \in S^1 : A(\theta, x) = x\}$.

Lemma 1 (Homomorphisms have trivial-or-surjective image)**.** *Any continuous homomorphism $\chi : S^1 \to \mathbb{T}_P \cong S^1$ is either trivial or surjective.*

Proof. The image of a connected set under a continuous map is connected. Connected subgroups of S^1 are either $\{0\}$ or S^1. □

Definition 8 (Nonzero slope)**.** *Nonzero slope means χ is nontrivial, equivalently surjective. Under the identifications $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}_P \cong \mathbb{R}/P\mathbb{Z}$, there exists $d \in \mathbb{Z}$ such that*

$$\chi(\theta) = \tfrac{P}{2\pi} d\,\theta \mod P.$$

Nonzero slope is $d \neq 0$.

Lemma 2 (Stabilizers are forced into the kernel)**.** *If ϕ is χ-equivariant, then for every $x \in \Sigma$,*

$$\mathrm{Stab}(x) \subseteq \ker \chi.$$

Proof. If $\theta \in \mathrm{Stab}(x)$, then $A(\theta, x) = x$, so equivariance gives $\phi(x) = \phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, hence $\chi(\theta) = 0$. □

3 Minimal forcing theorem

Theorem 13 (Circle-action forcing, minimal form)**.** *Let (Σ, A) be a slice with a continuous S^1-action and let $\phi : \Sigma \to \mathbb{T}_P$ be χ-equivariant. If χ has nonzero slope, then*

$$\phi(\Sigma) = \mathbb{T}_P.$$

Proof. Pick any $x \in \Sigma$. By Lemma 2, the orbit $O_x \cong S^1/\mathrm{Stab}(x)$ is compatible with χ. By equivariance, $\phi(A(\theta, x)) = \phi(x) \oplus \chi(\theta)$, so $\phi(O_x) = \phi(x) \oplus \chi(S^1) = \mathbb{T}_P$ since χ is surjective. As $O_x \subseteq \Sigma$, we have $\phi(\Sigma) = \mathbb{T}_P$. □

4 Upgrade: necessary and sufficient conditions

Theorem 14 (Slice-wise coverage under an equivariant circle action)**.** *Let* (Σ, A, ϕ, χ) *be as above, with* ϕ χ*-equivariant. Then* $\phi(\Sigma) = \mathbb{T}_P$ *holds if and only if either*

- χ *has nonzero slope, or*
- χ *is trivial and the induced map* $\bar{\phi} : \Sigma/S^1 \to \mathbb{T}_P$ *is surjective.*

In particular, in the nonzero-slope regime, coverage is automatic.

Proof. If χ has nonzero slope, Theorem 13 gives coverage. If χ is trivial, equivariance means ϕ is constant on each orbit and factors through Σ/S^1, so coverage is equivalent to surjectivity of the factor. □

5 Finite obstruction catalog (witness-first form)

Finite obstruction list is complete in the declared regime

Inside the declared regime, the obstruction interface is not an open-ended list of failure stories. It is a finite catalog whose members are forced by validator predicate failures. This is the sense in which the catalog is *complete*: every certificate failure inside the regime is recognized as exactly one canonical obstruction code with auditable evidence.

Canonical obstruction catalog (WOB)

WOB code	Meaning (declared regime)
`WOB-01-no-subsystem`	No admissible cyclic subsystem/observable is available in the measured model.
`WOB-02-trivial-winding`	All candidate loops have degree/holonomy multiple 0.
`WOB-03-pinch`	Image misses an arc neighborhood on the declared target (non-surjective).
`WOB-04-tear`	Regularity/definability needed for degree/holonomy fails at the declared tolerance.
`WOB-05-holonomy-defect`	Lift/connection consistency needed for holonomy certification fails.
`WOB-06-coupling-mismatch`	Coupling target/baseline is wrong (CST not computed or baseline not defined).
`WOB-07-mixing-not-justified`	Statistical convergence claim made without an admissible RNF or NDB+MXC hypothesis.
`WOB-08-orbit-collapse`	No χ-active circle-type orbit exists on the declared evidence domain.
`WOB-09-equivariance-defect`	Equivariance defect exceeds the declared admissible tolerance.
`WOB-10-resonant-locking`	Rank deficiency/rational dependence collapses the intended torus target.
`WOB-11-disconnected-slice`	Measurement domain misses the orbit/loop component required by the mechanism.
`WOB-12-slope-zero-quotient-nonsurjective`	χ is trivial and the induced quotient factor map is not surjective.

Theorem (catalog completeness within the declared regime)

Fix the declared regime described above (declared target type, admissible mechanism, admissible FEQ class, declared tolerances and margins, and in coupled settings the target-correctness rule selecting CST_s with coupled Haar baseline). Let *WSC* mean: *there exists a witness stability certificate satisfying the regime validators on the declared correct target.* Let *WOB* mean: *there exists a witness obstruction whose code is one of the fifteen canonical codes and whose evidence satisfies its checklist.* Then, inside the declared regime,

$$\neg\mathrm{WSC} \iff \mathrm{WOB}.$$

More concretely: a validator predicate failure $\neg\mathrm{V}_{XX}$ forces the obstruction code `WOB-XX`, and if all validator predicates hold then the regime permits a WSC on the correct target.

Necessity (each obstruction code is realized by an audited atlas artifact)

Each canonical WOB code occurs in the shipped witness atlas as a boundary artifact with concrete evidence. The following obstruction artifacts are included under `engine/artifacts/` and are checked by the audit suite:

WOB code	**Realizing atlas obstruction artifact**
`WOB-01-no-subsystem`	`WOB-ATLAS-0013.json`
`WOB-02-trivial-winding`	`WOB-ATLAS-0014.json`
`WOB-03-pinch`	`WOB-ATLAS-0015.json`
`WOB-04-tear`	`WOB-ATLAS-0016.json`
`WOB-05-holonomy-defect`	`WOB-ATLAS-0031.json`
`WOB-06-coupling-mismatch`	`WOB-ATLAS-0011.json`
`WOB-07-mixing-not-justified`	`WOB-ATLAS-0020.json`
`WOB-08-orbit-collapse`	`WOB-ATLAS-0019.json`
`WOB-09-equivariance-defect`	`WOB-ATLAS-0012.json`
`WOB-10-resonant-locking`	`WOB-ATLAS-0009.json`
`WOB-11-disconnected-slice`	`WOB-ATLAS-0017.json`
`WOB-12-slope-zero-quotient-nonsurjective`	`WOB-ATLAS-0018.json`

This realizability statement is what makes the catalog *necessary*: no code can be removed without deleting a genuine, audited boundary mechanism.

SFT phrases universality as a dichotomy: either a witness certificate exists (WSC), or a finite obstruction object exists (WOB). Write $\mathrm{SCL}^{\mathrm{SFT}}(\Sigma)$ for *witnessed coverage on the correct target* in the declared model class.

Predicate exhaustion lemma suite

Each obstruction code corresponds to a single validator predicate. Within the declared regime, if the predicate fails then the corresponding WOB code is forced and no WSC can exist without changing the declared target, mechanism, or model class.

Predicate exhaustion: minimal completeness skeleton

SFT treats *validator predicates* as the formal boundary between certificate-claims and obstruction-claims.

The audit-facing specification of the predicate suite and its canonical mapping appears in `VALIDATOR_PREDICATE_SPEC.tex` (and in the runnable contract files under `engine/contracts`). Inside the declared regime, a witness stability certificate (WSC) is permitted only when the listed predicates hold on the declared target. When a predicate fails, the correct output is a witness obstruction (WOB) with a canonical code and a finite evidence checklist.

Validator predicates and canonical WOB codes

Write V_{XX} for the validator predicate associated to obstruction code `WOB-XX`. The intended reading is:

- if V_{XX} holds, then `WOB-XX` is *not* triggered by that failure mode,
- if V_{XX} fails, then `WOB-XX` is the correct obstruction code to report (with evidence).

The predicates are:

- V_{01} **(subsystem/observable present).** There exists an admissible subsystem (slice, orbit-domain, or embedded loop/defect domain) and a phase observable to a declared target $\mathbb{T}_P$ or $\mathbb{T}_{\mathbf{P}}$. If this fails, report `WOB-01-no-subsystem`.
- V_{02} **(nontrivial winding/holonomy).** In an embedded mechanism, some certified loop carries a nonzero integer witness (degree or holonomy multiple). If this fails, report `WOB-02-trivial-winding`.
- V_{03} **(no pinch on the declared target).** On the declared target, the observable image does not miss a positive-length arc neighborhood required for SCL. If this fails, report `WOB-03-pinch`.
- V_{04} **(no tear: definability/regularity).** The hypotheses needed to define degree/holonomy in the adopted model class hold on the evidence domain (liftability/atlas consistency/continuity at the declared tolerance). If this fails, report `WOB-04-tear`.
- V_{05} **(holonomy consistency).** In holonomy-based mechanisms, the lift/connection interface is consistent at the declared tolerance so the holonomy integer is well-defined. If this fails, report `WOB-05-holonomy-defect`.
- V_{06} **(coupled target correctness).** In coupled settings, the selected target is the coupled subtorus coset CST_s and the baseline is coupled Haar on CST_s. If this fails, report `WOB-06-coupling-mismatch`.
- V_{07} **(mixing claim justified).** Any statistical convergence claim is accompanied by an admissible certificate hypothesis (RNF or NDB+MXC, with declared parameters). If this fails, report `WOB-07-mixing-not-justified`.
- V_{08} **(orbit realized on the evidence domain).** In action-equivariant mechanisms, the evidence domain contains a χ-active circle-type orbit segment needed for the forcing template. If this fails, report `WOB-08-orbit-collapse`.
- V_{09} **(equivariance within tolerance).** The equivariance defect is within the declared admissible tolerance on the evidence domain. If this fails, report `WOB-09-equivariance-defect`.

- V_{10} **(no resonant locking for the intended target).** The response rank and the inferred constraint structure do not collapse the intended target (full torus versus subtorus coset). If this fails, report `WOB-10-resonant-locking`.
- V_{11} **(connected slice/component availability).** The declared measurement domain intersects the orbit-bearing/loop-bearing component required by the chosen mechanism. If this fails, report `WOB-11-disconnected-slice`.
- V_{12} **(slope-zero quotient surjectivity when needed).** When the induced homomorphism χ is trivial, the factor map through the quotient is surjective onto the declared target. If this fails, report `WOB-12-slope-zero-quotient-nonsurjective`.
- V_{13} **(constraint hypothesis uniqueness).** In constraint/target inference, the declared data and tolerance determine a unique small-integer constraint hypothesis within the declared finite search. If this fails, report extttWOB-13-ambiguity.
- V_{14} **(sufficient excitation for inference).** The sample count and excitation are sufficient to support stable inference of constraint rank and target rank at the declared tolerance. If this fails, report extttWOB-14-insufficient-excitation.
- V_{15} **(stability margin away from resonance).** The best constraint candidates are separated from the declared tolerance by a positive stability margin; near-relations do not make the decision unstable. If this fails, report extttWOB-15-resonance-masking.

Lemma suite (each predicate failure forces its WOB code)

Lemma 3 (`WOB-01` is forced by $\neg\mathrm{V}_{01}$)**.** *If no admissible subsystem/observable exists in the declared regime, then no WSC exists in that regime.*

Proof. Every WSC is defined on a declared evidence domain and records a target and observable. If such data do not exist in the declared regime, then no WSC object satisfying the regime validators can exist. □

Lemma 4 (`WOB-02` is forced by $\neg\mathrm{V}_{02}$)**.** *If every admissible loop mechanism in the evidence domain has zero degree and zero holonomy multiple, then no embedded-loop WSC with nonzero witness integer exists.*

Proof. Embedded-loop WSCs require an integer witness invariant that persists under the LCM/HMR margins. If all admissible loops have witness integer 0, then the mechanism cannot certify forced coverage by winding/holonomy and the obstruction is the trivial-winding class. □

Lemma 5 (`WOB-03` is forced by $\neg\mathrm{V}_{03}$)**.** *If the observable image misses a required arc neighborhood on the declared target, then SCL on that target cannot hold, hence no target-correct WSC exists.*

Proof. SCL asserts surjectivity onto the declared target. Missing an arc neighborhood contradicts surjectivity, so no WSC proving SCL on that target can exist. □

Lemma 6 (`WOB-04` is forced by $\neg\mathrm{V}_{04}$)**.** *If the adopted model class does not provide the regularity needed to define the witness invariant (degree/holonomy) robustly at the declared tolerance, then no WSC with certified margins exists.*

Proof. The WSC format requires a well-defined integer witness together with margins certifying its stability. If the witness cannot be defined in the adopted model class on the evidence domain, the certificate interface cannot be satisfied. □

Lemma 7 (`WOB-05` is forced by $\neg \mathrm{V}_{05}$)**.** *If holonomy consistency fails at the declared tolerance, then holonomy-based certification is ill-posed in that regime and no holonomy WSC exists.*

Proof. Holonomy WSCs require a connection/lift interface whose loop integral is well-defined modulo the period. If the interface is inconsistent, the holonomy integer is not stable or not defined, and the holonomy-defect obstruction is the correct output. □

Lemma 8 (`WOB-06` is forced by $\neg \mathrm{V}_{06}$)**.** *In a coupled setting, if the target is not declared as* CST_s *with coupled Haar baseline, then any claimed certificate is target-incorrect and no target-correct WSC exists without repairing the target.*

Proof. Coupling changes the maximal invariant set and the correct baseline measure. A certificate that uses the wrong target/baseline is not a WSC in the declared target-correct regime. □

Lemma 9 (`WOB-07` is forced by $\neg \mathrm{V}_{07}$)**.** *If a statistical convergence claim is asserted without an admissible hypothesis certificate (RNF or NDB+MXC), then the claim is outside the permitted regime and the correct output is the mixing-not-justified obstruction.*

Proof. SFT treats convergence as an additional artifact class whose hypotheses must be declared and auditable. Without an admissible hypothesis certificate, no convergence certificate can be issued, so the regime boundary is triggered. □

Lemma 10 (`WOB-08` is forced by $\neg \mathrm{V}_{08}$)**.** *If the χ-active orbit needed by the equivariant forcing template is absent from the evidence domain, then action-equivariant forcing cannot be invoked there and no forcing-based WSC exists on that domain.*

Proof. The forcing template certifies coverage by transporting phases along a realized orbit. If no such orbit occurs on the evidence domain, the mechanism is unavailable and the orbit-collapse class is the correct obstruction. □

Lemma 11 (`WOB-09` is forced by $\neg \mathrm{V}_{09}$)**.** *If equivariance defect exceeds the declared tolerance, then equivariance is not valid in the adopted model class at that tolerance and no equivariance-based WSC exists.*

Proof. Equivariance-based WSCs require the defect to be within tolerance so that the induced homomorphism controls orbit images. If the defect is too large, the required mechanism hypotheses fail. □

Lemma 12 (`WOB-10` is forced by $\neg \mathrm{V}_{10}$)**.** *If rank deficiency or resonant locking collapses the intended target, then no certificate can truthfully assert coverage of the larger target; the correct output is the resonant-locking obstruction (and the target must be revised).*

Proof. The correct target is determined by the response rank and by any valid integer constraints. If the intended target is larger than the reachable coset/subtorus, coverage of the larger target is impossible. □

Lemma 13 (`WOB-11` is forced by $\neg \mathrm{V}_{11}$)**.** *If the declared measurement domain does not meet the required orbit-bearing/loop-bearing component, then no certificate for that mechanism can be produced from that domain.*

Proof. All certificate mechanisms require evidence on a domain supporting the relevant orbit or loop. If the domain misses that component, the mechanism is inapplicable there. □

Lemma 14 (`WOB-12` is forced by $\neg V_{12}$)**.** *If χ is trivial and the quotient factor map is not surjective onto the declared target, then SCL on that target is impossible and no target-correct WSC exists.*

Proof. When χ is trivial, equivariance forces the observable to factor through the quotient. If the induced factor is not surjective, the declared target cannot be covered. □

Lemma 15 (extttWOB-13 is forced by $eg\mathrm{V}_{13}$)**.** *If multiple incompatible small-integer constraint hypotheses fit within the declared tolerance and search, then the target is not uniquely determined in the declared regime and no target-correct certificate can be issued without additional excitation or a refined target declaration.*

Proof. A target-correct certificate requires a declared target type and, when constraints are part of the target selection, a determinate constraint hypothesis. If the declared data/tolerance admit more than one incompatible constraint hypothesis within the search, any choice is non-auditable without additional information, so the declared regime treats this as an explicit inference obstruction. □

Lemma 16 (extttWOB-14 is forced by $eg\mathrm{V}_{14}$)**.** *If the sample count or excitation is insufficient to support stable constraint inference at the declared tolerance, then inference is ill-posed in the declared regime and the correct output is extttWOB-14-insufficient-excitation.*

Proof. Stable inference requires enough samples and enough motion to distinguish candidate constraint hypotheses. When these are absent, any inferred constraints can change under small perturbations of the data, so no target-correct certificate can be issued in the declared regime. □

Lemma 17 (extttWOB-15 is forced by $eg\mathrm{V}_{15}$)**.** *If the best candidate constraints are within the declared tolerance by too small a margin, or if a near-relation lies just above tolerance, then the constraint decision is unstable at the declared tolerance and the correct output is extttWOB-15-resonance-masking.*

Proof. A positive stability margin is needed so that the inferred constraints persist under admissible perturbations. When the margin is too small or a near-relation is present, small perturbations can change the inferred target rank or constraint set, so the declared regime treats the situation as a resonance-masking obstruction. □

Validator predicate specification (audit interface)

The completeness claim in the declared regime is enforced by a finite suite of *validator predicates*. Each predicate V_{XX} corresponds to one canonical obstruction code `WOB-XX`. For an issued WSC, the applicable predicates must hold; otherwise the correct output is a WOB with the associated code.

For audit-level synchronization between the paper and the runnable artifacts, the normative mapping is also recorded in:

- `engine/contracts/VALIDATOR_PREDICATE_SPEC.md`
- `engine/contracts/validator_predicates.json`
- `engine/contracts/wob_codebook.json`

Pred.	Canonical WOB code	Applicability (minimal)	Audit hook (artifact-level)
V01	`WOB-01-no-subsystem`	always	`audit_wsc.py`: requires a nonempty target and a predicate ledger entry for V01
V02	`WOB-02-trivial-winding`	mechanisms LDW/HW	`audit_wsc.py`: requires witness integer $k \neq 0$ and a V02 ledger entry
V03	`WOB-03-pinch`	when arc-neighborhood coverage is asserted	recorded as an evidence reference (no recomputation in v0.1)
V04	`WOB-04-tear`	mechanisms LDW/HW	`audit_wsc.py`: requires declared perturbation model + margins and a V04 ledger entry
V05	`WOB-05-holonomy-defect`	when holonomy multiple is invoked	`audit_wsc.py`: requires `margins.HMR_P` and a V05 ledger entry
V06	`WOB-06-coupling-mismatch`	target is CST (or mechanism CMT)	CTP/CHT (coupling target correctness) enforced: requires `target_type=cst` and baseline declaring coupled Haar on CST
V07	`WOB-07-mixing-not-justified`	mechanism MXC	`audit_wsc.py`: requires a convergence certificate (RNF or NDB+MXC) and a V07 ledger entry
V08	`WOB-08-orbit-collapse`	mechanism AWT	`audit_wsc.py`: requires a V08 ledger entry with evidence reference
V09	`WOB-09-equivariance-defect`	mechanism AWT	`audit_wsc.py`: requires a V09 ledger entry with evidence reference
V10	`WOB-10-resonant-locking`	when full-torus target is asserted	recorded as an evidence reference (rank/constraint inference expands in SFT III)
V11	`WOB-11-disconnected-slice`	when component restriction is used	recorded as an evidence reference (connectedness tests expand with payloads)
V12	`WOB-12-slope-zero-quotient-nonsurjective`	when χ is trivial and quotient is invoked	recorded as an evidence reference (quotient surjectivity expands with payloads)
V13	`WOB-13-ambiguity`	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)

Pred.	Canonical WOB code	Applicability (minimal)	Audit hook (artifact-level)
V14	`WOB-14-insufficient-excitation`	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)
V15	`WOB-15-resonance-masking`	when CST/constraint inference is invoked	recorded as a benchmark/tool evidence reference (no recomputation in v0.1)

The WSC artifact format includes a lightweight predicate ledger field `validator_predicates` that records the applicable predicates as "holds" with short evidence references. The audit layer checks the parts that are readable from the artifact interface and rejects target-incorrect coupled claims.

6 Robustness (quantitative)

Use the geodesic metric on $\mathbb{T}_P$, $d_{\mathbb{T}_P}([u],[v]) = \min_{n\in\mathbb{Z}} |u - v + nP|$. Define the equivariance defect on a domain $D \subseteq \Sigma$ by

$$\mathrm{Def}_D(A,\phi,\chi) := \sup_{\theta\in S^1,\ x\in D} d_{\mathbb{T}_P}(\phi(A(\theta,x)), \phi(x) \oplus \chi(\theta)).$$

Theorem 15 (Robust forcing on an orbit)**.** *Assume χ has nonzero slope and $\mathrm{Def}_D(A,\phi,\chi) \le \varepsilon$. Then for any $x \in D$, the image of ϕ on the orbit segment $O_x \cap D$ is ε-dense in $\mathbb{T}_P$. Moreover, if on some orbit $O \cong S^1$ the restriction admits a lift $f(t) = \frac{dP}{2\pi}t + \psi(t)$ with $\|\psi'\|_\infty < \frac{|d|P}{2\pi}$, then $\phi(O) = \mathbb{T}_P$.*

7 Multi-cycle forcing and coupled targets

Let $\mathbb{T}_{\mathbf{P}} := \prod_{j=1}^{k} \mathbb{T}_{P_j}$. A $\mathbb{T}^k$-equivariant observable $\Phi : \Sigma \to \mathbb{T}_{\mathbf{P}}$ is governed by a homomorphism $\chi : \mathbb{T}^k \to \mathbb{T}_{\mathbf{P}}$, represented (after standard identifications) by an integer matrix $D \in \mathbb{Z}^{k\times k}$. Write $r := \mathrm{rank}(D)$.

Theorem 16 (Torus forcing, correct target form)**.** *If Φ is χ-equivariant, then for every $x \in \Sigma$,*

$$\Phi(\mathbb{T}^k \cdot x) = \Phi(x) \oplus \mathrm{Im}(\chi),$$

a coset of the subtorus $\mathrm{Im}(\chi) \subseteq \mathbb{T}_{\mathbf{P}}$. If the $\mathbb{T}^k$-action is transitive on a slice Σ_s (so Σ_s is a $\mathbb{T}^k$-torsor), then $\Phi(\Sigma_s)$ is a single coset of $\mathrm{Im}(\chi)$; after choosing a basepoint normalization it may be identified with $\mathrm{Im}(\chi)$. In particular, if $r = k$ then $\mathrm{Im}(\chi) = \mathbb{T}_{\mathbf{P}}$ and every coset is the full torus, while if $r < k$ then every orbit-image is an r-dimensional subtorus coset.

If coupling constraints restrict the image to a coupled subtorus $\mathrm{CST}_s \subseteq \mathbb{T}_{\mathbf{P}}$, the correct target is $\Phi(\Sigma_s) = \mathrm{CST}_s$ and the correct statistical baseline is Haar on CST_s.

8 Ergodic universality

Theorem 17 (Unique ergodicity of irrational torus translations)**.** *If $R_\omega : \mathbb{T}^k \to \mathbb{T}^k$ is translation by $\omega \in \mathbb{R}^k$ with $1, \omega_1, \ldots, \omega_k$ linearly independent over $\mathbb{Q}$, then R_ω is uniquely ergodic and time averages converge to Haar.*

9 Dynamics preservation and diffusion

If dynamics commute with the symmetry action and preserve the invariants used by the witness certificates (degree/holonomy/coupled relation), then witnesses persist in time. On tori, diffusion drives distributions exponentially fast to Haar; under standard irreducibility assumptions for advection–diffusion, convergence is exponential in total variation.

10 Minimal structure and sharp boundary

For the forcing implication, the only required ingredients are a sliced arena, a nontrivial cyclic symmetry action, and an equivariant phase observable with nonzero slope. If a slice admits no admissible cyclic symmetry (no nontrivial S^1 subgroup in the admissible automorphism class), the action-forcing witness engine cannot apply; any Syncré claim must be embedded-witness based or must add structure.

11 Capstone global theorem

Theorem 18 (Global Syncré universality)**.** *Let $M = \bigsqcup_s \Sigma_s$ be a sliced arena. Suppose that on each slice:*

- *there is admissible cyclic symmetry (an S^1 or $\mathbb{T}^k$ action),*
- *there exists a phase field that is equivariant with nonzero slope, with no degree/holonomy obstruction in the adopted model class,*
- *in coupled settings, the correct target CST_s is selected and witness claims are made on that target.*

Then slice-wise coverage holds on the correct target (witness-first SCL), is robust under admissible perturbations preserving the obstruction-free regime, is preserved under admissible dynamics, converges to the correct Haar target under nonresonant drives, and converges exponentially under diffusion/noise. Failures are classified by the finite WOB obstruction catalog.

Capstone theorem (COC $\Rightarrow$ WSC/WOB)

Theorem 19 (COC engine dichotomy)**.** *Fix the declared hypothesis regime (including target type and perturbation model). For every context in the Cyclic Observability Class (COC), the SFT engine outputs exactly one of the following:*

- *a Witness Stability Certificate (WSC) asserting coverage on the correct target, with explicit margins and a validator predicate ledger, or*
- *a Witness Obstruction (WOB) whose canonical code identifies the first failed validator predicate, together with evidence.*

Any statistical or mixing statement is an optional upgrade: it appears only when a convergence certificate is explicitly declared and its baseline matches the correct target.

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